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ELEMENTS

OF THE

INTEGRAL CALCULUS,

WITH A

KEY TO THE SOLUTION OF DIFFERENTIAL EQUATIONS.

BY

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PREFACE.

THE following volume is a sequel to my treatise on the Differential Calculus, and, like that, is written as a text-book. The last chapter, however, a Key to the Solution of Differential Equations, may prove of service to working mathematicians.

I have used freely the works of Bertrand, Benjamin Peirce, Todhunter, and Boole; and I am much indebted to Professor J. M. Peirce for criticisms and suggestions.

I refer constantly to my work on the Differential Calculus as Volume I.; and for the sake of convenience I have added Chapter V. of that book, which treats of Integration, as an appendix to the present volume.

W. E. BYERLY.

CAMBRIDGE, 1881.

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INTEGRAL CALCULUS.

CHAPTER I.

SYMBOLS OF OPERATION.

1. It is often convenient to regard a functional symbol as indicating *an operation to be performed upon the expression which is written after the symbol.* From this point of view the symbol is called a *symbol of operation*, and the expression written after the symbol is called the *subject* of the operation.

Thus the symbol D_x in $D_x(x^2y)$ indicates that the *operation* of differentiating with respect to x is to be performed upon the *subject* (x^2y).

2. If the *result* of one operation is taken as the *subject* of a second, there is formed what is called a *compound function.*

Thus $\log\sin x$ is a *compound function*, and we may speak of the taking of the $\log\sin$ as a *compound operation.*

3. When two operations are so related that the compound operation, in which the result of performing the first on any subject is taken as the subject of the second, leads to the same result as the compound operation, in which the result of performing the second on the same subject is taken as the subject of the first, the two operations are *commutative* or *relatively free.*

Or to formulate; if

$$fFu = Ffu,$$

the operations indicated by f and F are *commutative.*

For example; the operations of partial differentiation with respect to two independent variables x and y are commutative, for we know that

$$D_x D_y u = D_y D_x u. \qquad \text{(I. Art. 197).}$$

The operations of taking the sine and of taking the logarithm are not commutative, for $\log \sin u$ is not equal to $\sin \log u$.

4. If $$f(u \pm v) = fu \pm fv$$

where u and v are any subjects, the operation f is *distributive* or *linear*.

The operation indicated by d and the operation indicated by D_x are distributive, for we know that

$$d(u \pm v) = du \pm dv,$$

and that $$D_x(u \pm v) = D_x u \pm D_x v.$$

The operation sin is not distributive, for $\sin(u+v)$ is not equal to $\sin u + \sin v$.

5. *The compounds of distributive operations are distributive.*

Let f and F indicate distributive operations, then fF will be distributive; for

$$F(u \pm v) = Fu \pm Fv,$$

therefore $$fF(u \pm v) = f(Fu \pm Fv) = fFu \pm fFv.$$

6. The *repetition* of any operation is indicated by writing an *exponent, equal to the number of times the operation is performed*, after the symbol of the operation.

Thus $\log^3 x$ means $\log\log\log x$; $d^3 u$ means $dddu$.

In the single case of the trigonometric functions a different use of the exponent is sanctioned by custom, and $\sin^2 u$ means $(\sin u)^2$ and not $\sin \sin u$.

7. If m and n are *whole numbers* it is easily proved that

$$f^m f^n u = f^{m+n} u.$$

This formula is assumed for all values of m *and* n, *and negative and fractional exponents are interpreted by its aid.* It is called the *law of indices.*

8. To find what interpretation must be given to a *zero exponent*, let

$$m = 0 \qquad \text{in the formula of Art. 7.}$$

$$f^0 f^n u = f^{0+n} u = f^n u,$$

or, denoting $f^n u$ by v, $\quad f^0 v = v.$

That is; *a symbol of operation with the exponent zero has no effect on the subject*, and may be regarded as multiplying it by unity.

9. To interpret a *negative exponent,* let

$$m = -n \qquad \text{in the formula of Art. 7.}$$

$$f^{-n} f^n u = f^{-n+n} u = f^0 u = u.$$

If we call $\quad f^n u = v$, then $f^{-n} v = u.$

If $\quad n = 1$

we get $\quad f^{-1} f u = u,$

and the exponent -1 indicates what we have called the anti-function of fu. (I. Art. 72.)

The exponent -1 is used in this sense even with trigonometric functions.

10. When two operations are *commutative* and *distributive*, the symbols which represent them may be combined precisely as if they were algebraic quantities.

For they obey the laws,

$$a(m + n) = am + an,$$

$$am = ma,$$

on which all the operations of arithmetic and algebra are founded.

For example; if the operation $(D_x + D_y)$ is to be performed n times in succession on a subject u, we can expand $(D_x + D_y)^n$ precisely as if it were a binominal, and then perform on u the operations indicated by the expanded expression.

$$\begin{aligned}(D_x + D_y)^3 u &= (D_x^3 + 3\,D_x^2 D_y + 3\,D_x D_y^2 + D_y^3)u \\ &= D_x^3 u + 3\,D_x^2 D_y u + 3\,D_x D_y^2 u + D_y^3 u.\end{aligned}$$

CHAPTER II.

IMAGINARIES.

11. An *imaginary* is usually defined in algebra as ***the indicated even root of a negative quantity***, and although it is clear that there can be no *quantity* that raised to an even power will be negative, the assumption is made that an imaginary can be treated like any algebraic quantity.

Imaginaries are first forced upon our notice in connection with the subject of quadratic equations. Considering the typical quadratic

$$x^2 + ax + b = 0,$$

we find that it has two roots, and that these roots possess certain important properties. For example; their sum is $-a$ and their product is b. We are led to the conclusion that every quadratic has two roots whose sum and whose product are simply related to the coefficients of the equation.

On trial, however, we find that there are quadratics having but one root, and quadratics having no root.

For example; if we solve the equation

$$x^2 - 2x + 1 = 0,$$

we find that the only value of x which will satisfy it is *unity;* and if we attempt to solve

$$x^2 - 2x + 2 = 0,$$

we find that there is no value of x which will satisfy the equation.

As these results are apparently inconsistent with the conclusion to which we were led on solving the general equation, we naturally endeavor to reconcile them with it.

The difficulty in the case of the equation which has but one

root is easily overcome by regarding it as having two equal roots. Thus we can say that each of the two roots of the equation

$$x^2 - 2x + 1 = 0$$

is equal to 1; and there is a decided advantage in looking at the question from this point of view, for the roots of this equation will possess the same properties as those of a quadratic having unequal roots. The sum of the roots 1 and 1 is minus the coefficient of x in the equation, and their product is the constant term.

To overcome the difficulty presented by the equation which has no root we are driven to the conception of *imaginaries.*

12. *An imaginary is not a quantity, and the treatment of imaginaries is purely arbitrary and conventional.* We begin by laying down a few arbitrary rules for our imaginary expressions to obey, which must not involve any contradiction; and we must perform all our operations upon imaginaries, and must interpret all our results by the aid of these rules.

Since imaginaries occur as roots of equations, they bear a close analogy with ordinary algebraic quantities, and they have to be subjected to the same operations as ordinary quantities; therefore our rules ought to be so chosen that the results may be comparable with the results obtained when we are dealing with real quantities.

13. By adopting the convention that

$$\sqrt{-a^2} = a\sqrt{-1},$$

where a is supposed to be *real*, we can reduce all our imaginary algebraic expressions to forms where $\sqrt{-1}$ is the only peculiar symbol. This symbol $\sqrt{-1}$ we shall define and use as *the symbol of some operation, at present unknown, the repetition of which has the effect of changing the sign of the subject of the operation.* Thus in $a\sqrt{-1}$ the symbol $\sqrt{-1}$ indicates that an operation is performed upon a which, if repeated, will change the sign of a. That is,

$$a(\sqrt{-1})^2 = -a.$$

From this point of view it would be more natural to write the symbol before instead of after the subject on which it operates, $(\sqrt{-1})a$ instead of $a\sqrt{-1}$, and this is sometimes done; but as the usage of mathematicians is overwhelmingly in favor of the second form, we shall employ it, merely as a matter of convenience, and remembering that a is the *subject* and the $\sqrt{-1}$ the *symbol of operation.*

14. The rules in accordance with which we shall use our new symbol are, first,

$$a\sqrt{-1}+b\sqrt{-1}=(a+b)\sqrt{-1}. \qquad [1]$$

In other words, the operation indicated by $\sqrt{-1}$ is to be *distributive* (Art. 4); and second,

$$a\sqrt{-1}=(\sqrt{-1})a, \qquad [2]$$

or our symbol is to be *commutative* with the symbols of quantity (Art. 3).

These two conventions will enable us to use our symbol in algebraic operations precisely as if it were a quantity (Art. 10).

When no coefficient is written before $\sqrt{-1}$ the coefficient 1 will be understood, or unity will be regarded as the subject of the operation.

15. Let us see what interpretation we can get for powers of $\sqrt{-1}$; that is, for repetitions of the operation indicated by the symbol.

$$\begin{aligned}
&(\sqrt{-1})^0=1 && \text{(Art. 8)},\\
&(\sqrt{-1})^1=\sqrt{-1}, &&\\
&(\sqrt{-1})^2=-1, && \text{by definition (Art. 13)},\\
&(\sqrt{-1})^3=(\sqrt{-1})^2\sqrt{-1}=-\sqrt{-1}, && \text{by definition},\\
&(\sqrt{-1})^4=-(\sqrt{-1})^2 \qquad =1, &&\\
&(\sqrt{-1})^5=1\sqrt{-1} \qquad =\sqrt{-1}, &&\\
&(\sqrt{-1})^6=(\sqrt{-1})^2 \qquad =-1, &&
\end{aligned}$$

and so on, the values $\sqrt{-1}$, -1, $-\sqrt{-1}$, 1, occurring in

cycles of four. We can formulate this as follows; let n be zero or any positive whole number, then,

$$(\sqrt{-1})^{4n} = 1,$$
$$(\sqrt{-1})^{4n+1} = \sqrt{-1},$$
$$(\sqrt{-1})^{4n+2} = -1,$$
$$(\sqrt{-1})^{4n+3} = -\sqrt{-1}.$$

16. The definition we have given for the square root of a negative quantity, and the rules we have adopted concerning its use, enable us to remove entirely the difficulty felt in dealing with a quadratic which does not have real roots. Take the equation

$$x^2 - 2x + 5 = 0. \tag{1}$$

Solving by the usual method, we get

$$x = 1 \pm \sqrt{-4};$$
$$\sqrt{-4} = 2\sqrt{-1}, \text{ by Art. 13 [1]};$$

hence $$x = 1 + 2\sqrt{-1} \text{ or } 1 - 2\sqrt{-1}.$$

On substituting these results in turn in the equation (1), performing the operations by the aid of our conventions (Art. 14 [1] and [2]), and interpreting $(\sqrt{-1})^2$ by Art. 15, we find that they both satisfy the equation, and that they can therefore be regarded as entirely analogous to real roots. We find, too, that their sum is 2 and that their product is 5, and consequently that they bear the same relations to the coefficients of the equation as real roots.

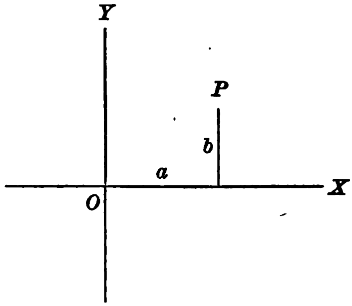
17. An imaginary root of a quadratic can always be reduced to the form $a + b\sqrt{-1}$ where a and b are real, and this is taken as the general type of an imaginary; and part of our work will be to show that when we subject imaginaries to the ordinary functional operations, all our results are reducible to this typical form.

18. We have defined $\sqrt{-1}$ as the symbol of an operation whose repetition changes the sign of the subject.

Several different interpretations of this operation have been suggested, and the following one, in which every imaginary is graphically represented by the position of a point in a plane, is commonly adopted, and is found exceedingly useful in suggesting and interpreting relations between different imaginaries and between imaginaries and reals.

In the *Calculus of Imaginaries*, $a+b\sqrt{-1}$ is taken as the general symbol of quantity. If b is equal to zero, $a+b\sqrt{-1}$ reduces to a, and is *real;* if a is equal to zero, $a+b\sqrt{-1}$ reduces to $b\sqrt{-1}$, and is called a *pure imaginary.*

$a+b\sqrt{-1}$ is represented by the position of a point referred to a pair of rectangular axes, as in analytic geometry, a being taken as the abscissa of the point and b as its ordinate. Thus in the figure the position of the point P represents the imaginary $a+b\sqrt{-1}$.



If $$b=0$$

and our quantity is real, P will lie on the axis of X, which on that account is called the *axis of reals;* if
$$a=0,$$

and we have a *pure imaginary*, P will lie on the axis of Y, which is called the *axis of pure imaginaries.*

Since a and $a\sqrt{-1}$ are represented by points equally distant from the origin, and lying on the *axis of reals* and the *axis of pure imaginaries* respectively, we may regard the operation indicated by $\sqrt{-1}$ as causing the point representing the subject of the operation to rotate about the origin through an angle of 90°. A repetition of the operation ought to cause the point to rotate 90° further, and it does; for

$$a(\sqrt{-1})^2=-a,$$

and is represented by a point at the same distance from the

origin as a, and lying on the opposite side of the origin; again repeat the operation,

$$a(\sqrt{-1})^3 = -a\sqrt{-1},$$

and the point has rotated 90° further; repeat again,

$$a(\sqrt{-1})^4 = a,$$

and the point has rotated through 360°. We see, then, that if the subject is a *real* or a *pure imaginary* the effect of performing on it the operation indicated by $\sqrt{-1}$ is to rotate it about the origin through the angle 90°. We shall see later that even when the subject is neither a real nor a pure imaginary, the effect of operating on it with $\sqrt{-1}$ is still to produce the rotation just described.

19. The *sum*, the *product*, and the *quotient* of any two imaginaries, $a + b\sqrt{-1}$ and $c + d\sqrt{-1}$, are imaginaries of the typical form.

$$a + b\sqrt{-1} + c + d\sqrt{-1} \quad = a + c + (b + d)\sqrt{-1}. \qquad [1]$$

$$(a + b\sqrt{-1})\,(c + d\sqrt{-1}) = ac - bd + (bc + ad)\sqrt{-1}. \qquad [2]$$

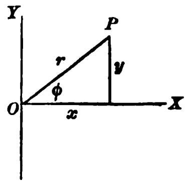
$$\frac{a+b\sqrt{-1}}{c+d\sqrt{-1}} = \frac{(a+b\sqrt{-1})\,(c-d\sqrt{-1})}{(c+d\sqrt{-1})\,(c-d\sqrt{-1})} = \frac{ac+bd+(bc-ad)\sqrt{-1}}{c^2+d^2}$$

$$= \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}\sqrt{-1}. \qquad [3]$$

All these results are of the form $A + B\sqrt{-1}$.

20. The graphical representation we have suggested for imaginaries suggests a second typical form for an imaginary. Given the imaginary $x + y\sqrt{-1}$, let the *polar coördinates* of the point P which represents $x + y\sqrt{-1}$ be r and ϕ.

r is called the *modulus* and ϕ the *argument* of the imaginary.



The figure enables us to establish very simple relations between x, y, r, and ϕ.

$$\left.\begin{aligned} x &= r\cos\phi, \\ y &= r\sin\phi; \end{aligned}\right\} \qquad [1]$$

$$\left.\begin{aligned} r &= \sqrt{x^2+y^2}, \\ \phi &= \tan^{-1}\frac{y}{x}. \end{aligned}\right\} \qquad [2]$$

$$\begin{aligned} x+y\sqrt{-1} &= r\cos\phi + (\sqrt{-1})r\sin\phi \\ &= r(\cos\phi + \sqrt{-1}.\sin\phi), \qquad [3] \end{aligned}$$

where the imaginary is expressed in terms of its modulus and argument.

The value of r given by our formulas [2] is ambiguous in sign; and ϕ may have any one of an infinite number of values differing by multiples of π. In practice we always take the positive value of r, and a value of ϕ which will bring the point in question into the right quadrant. In the case of any given imaginary then, r can have but one value, while ϕ may have any one of an infinite number of values differing by 2π.

EXAMPLES.

(1) Find the modulus and argument of 1; of $\sqrt{-1}$; of -4; of $-2\sqrt{-1}$; of $3+3\sqrt{-1}$; of $2+4\sqrt{-1}$; and express each of these quantities in the form $r(\cos\phi+\sqrt{-1}.\sin\phi)$.

(2) Show that every positive real has the argument zero; every negative real the argument π; every positive pure imaginary the argument $\frac{\pi}{2}$; and every negative pure imaginary the argument $\frac{3\pi}{2}$.

21. If we add two imaginaries, the *modulus of the sum* is never greater than the *sum of the moduli* of the given imaginaries.

The sum of $a+b\sqrt{-1}$ and $c+d\sqrt{-1}$ is $a+c+(b+d)\sqrt{-1}$. The modulus of this sum is $\sqrt{(a+c)^2+(b+d)^2}$; the sum of the moduli of $a+b\sqrt{-1}$ and $c+d\sqrt{-1}$ is $\sqrt{a^2+b^2}+\sqrt{c^2+d^2}$. We wish to show that

$$\sqrt{(a+c)^2+(b+d)^2} \prec \sqrt{a^2+b^2}+\sqrt{c^2+d^2};$$

the sign $\prec$ meaning "*equal to or less than.*"

Now $$\sqrt{(a+c)^2+(b+d)^2} \prec \sqrt{a^2+b^2}+\sqrt{c^2+d^2},$$

if $(a+c)^2+(b+d)^2 \prec a^2+b^2+2\sqrt{(a^2+b^2)(c^2+d^2)}+c^2+d^2$,

that is, if $$ac+bd \prec \sqrt{a^2c^2+a^2d^2+b^2c^2+b^2d^2};$$

or, squaring, if

$$a^2c^2+2abcd+b^2d^2 \prec a^2c^2+a^2d^2+b^2c^2+b^2d^2;$$

or, if $$0 \prec (ad-bc)^2.$$

This last result is necessarily true, as no real can have a square less than zero; hence our proposition is established.

22. *The modulus of the product of two imaginaries is the product of the moduli of the given imaginaries, and the argument of the product is the sum of the arguments of the imaginaries.*

Let us multiply

$$r_1(\cos\phi_1+\sqrt{-1}.\sin\phi_1) \quad \text{by} \quad r_2(\cos\phi_2+\sqrt{-1}.\sin\phi_2);$$

we get

$$r_1r_2[\cos\phi_1\cos\phi_2-\sin\phi_1\sin\phi_2+\sqrt{-1}(\sin\phi_1\cos\phi_2+\cos\phi_1\sin\phi_2)],$$

$$\cos\phi_1\cos\phi_2-\sin\phi_1\sin\phi_2=\cos(\phi_1+\phi_2),$$

$$\sin\phi_1\cos\phi_2+\cos\phi_1\sin\phi_2=\sin(\phi_1+\phi_2)$$

by Trigonometry; hence

$$r_1(\cos\phi_1+\sqrt{-1}.\sin\phi_1)\,r_2(\cos\phi_1+\sqrt{-1}.\sin\phi_2)$$
$$=r_1r_2[\cos(\phi_1+\phi_2)+\sqrt{-1}.\sin(\phi_1+\phi_2)],$$

and our result is in the typical form, $r_1 r_2$ being the modulus and $\phi_1 + \phi_2$ the argument of the product.

If each factor has the modulus unity, this theorem enables us to construct very easily the product of the imaginaries; it also enables us to show that the interpretation of the operation $\sqrt{-1}$, suggested in Art. 18, is perfectly general.

Let us operate on any imaginary subject,

$$r(\cos\phi + \sqrt{-1}.\sin\phi), \qquad \text{with } \sqrt{-1},$$

that is, with

$$1\left(\cos\frac{\pi}{2} + \sqrt{-1}.\sin\frac{\pi}{2}\right).$$

The modulus r will be unchanged, the argument ϕ will be increased by $\frac{\pi}{2}$, and the effect will be to cause the point representing the given imaginary to rotate about the origin through an angle of 90°.

23. Since division is the inverse of multiplication,

$$r_1(\cos\phi_1 + \sqrt{-1}.\sin\phi_1) \div r_2(\cos\phi_2 + \sqrt{-1}.\sin\phi_2)$$

will be equal to

$$\frac{r_1}{r_2}[\cos(\phi_1 - \phi_2) + \sqrt{-1}.\sin(\phi_1 - \phi_2)],$$

since if we multiply this by $r_2(\cos\phi_2 + \sqrt{-1}.\sin\phi_2)$, according to the method established in Art. 22, we must get

$$r_1(\cos\phi_1 + \sqrt{-1}.\sin\phi_1).$$

To divide one imaginary by another, we have then to take the quotient obtained by dividing the modulus of the first by the modulus of the second as our required modulus, and the argument of the first minus the argument of the second as our new argument.

24. If we are dealing with the product of n equal factors, or, in other words, if we are raising $r(\cos\phi + \sqrt{-1}.\sin\phi)$ to the

nth power, n being a positive whole number, we shall have, by Art. 22,

$$[r(\cos\phi + \sqrt{-1}.\sin\phi)]^n = r^n(\cos n\phi + \sqrt{-1}.\sin n\phi). \quad [1]$$

If r is unity, we have merely to multiply the argument by n, without changing the modulus; so that in this case increasing the exponent by unity amounts to rotating the point representing the imaginary through an angle equal to ϕ without changing its distance from the origin.

25. Since extracting a root is the inverse of raising to a power,

$$\sqrt[n]{[r(\cos\phi + \sqrt{-1}.\sin\phi)]} = \sqrt[n]{r}\left(\cos\frac{\phi}{n} + \sqrt{-1}.\sin\frac{\phi}{n}\right); \quad [1]$$

for, by Art. 24,

$$\left[\sqrt[n]{r}\left(\cos\frac{\phi}{n} + \sqrt{-1}.\sin\frac{\phi}{n}\right)\right]^n = r\,(\cos\phi + \sqrt{-1}.\sin\phi).$$

EXAMPLE.

Show that Art. 24 [1] holds even when n is negative or fractional.

26. As the *modulus of every quantity*, positive, negative, real, or imaginary, *is positive*, it is always possible to find the modulus of any required root; and as this modulus must be real and positive, *it can never*, in any given example, *have more than one value*. We know from algebra, however, that every equation of the nth degree containing one unknown has n roots, and that consequently every number must have n nth roots. Our formula, Art. 25 [1], appears to give us but one nth root for any given quantity. It must then be incomplete.

We have seen (Art. 20) that while the modulus of a given imaginary has but one value, its argument is indeterminate and may have any one of an infinite number of values which differ by multiples of 2π. If ϕ_0 is one of these values, the full form of

the imaginary is not $r(\cos\phi_0 + \sqrt{-1}.\sin\phi_0)$, as we have hitherto written it, but is

$$r[\cos(\phi_0 + 2m\pi) + \sqrt{-1}.\sin(\phi_0 + 2m\pi)],$$

where m is zero or any whole number positive or negative. Since angles differing by multiples of 2π have the same trigonometric functions, it is easily seen that the introduction of the term $2m\pi$ into the argument of an imaginary will not modify any of our results except that of Art. 25, which becomes

$$\sqrt[n]{r[\cos(\phi_0 + 2m\pi) + \sqrt{-1}.\sin(\phi_0 + 2m\pi)]}$$
$$= \sqrt[n]{r}\left[\cos\left(\frac{\phi_0}{n} + m\frac{2\pi}{n}\right) + \sqrt{-1}.\sin\left(\frac{\phi_0}{n} + m\frac{2\pi}{n}\right)\right]. \quad [1]$$

Giving m the values $0, 1, 2, 3 \ldots\ldots, n-1, n, n+1$, successively, we get

$$\frac{\phi_0}{n},\ \frac{\phi_0}{n} + \frac{2\pi}{n},\ \frac{\phi_0}{n} + 2\frac{2\pi}{n},\ \frac{\phi_0}{n} + 3\frac{2\pi}{n} \ldots\ldots,\ \frac{\phi_0}{n} + \overline{n-1}\,\frac{2\pi}{n},$$
$$\frac{\phi_0}{n} + 2\pi,\ \frac{\phi_0}{n} + \frac{2\pi}{n} + 2\pi,$$

as arguments of our nth root.

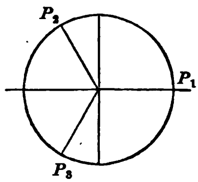
Of these values the first n, that is, all except the last two, correspond to different points, and therefore to different roots; the next to the last gives the same point as the first, and the last the same point as the second, and it is easily seen that if we go on increasing m we shall get no new points. The same thing is true of negative values of m.

Hence we see that *every quantity, real or imaginary, has* n *distinct* n*th roots, all having the same modulus*, but with arguments differing by multiples of $\frac{2\pi}{n}$.

27. Any *positive real* differs from unity only by its modulus, and any *negative real* differs from -1 only by its modulus. All the nth roots of any number or of its negative may be obtained

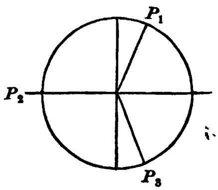
by multiplying the nth roots of 1 or of -1 by the real positive nth roots of the number.

Let us consider some of the roots of 1 and of -1; for example, the cube roots of 1 and of -1. The modulus of 1 is 1, and its argument is 0. The modulus of each of the cube roots of 1 is 1, and their arguments are 0, $\frac{2\pi}{3}$, and $\frac{4\pi}{3}$; that is, $0°$, $120°$, and $240°$. The roots in question, then, are represented by the points P_1, P_2, P_3, in the figure. Their values are



$$1(\cos 0 + \sqrt{-1}.\sin 0),$$
$$1(\cos 120° + \sqrt{-1}.\sin 120°),$$
and $$1(\cos 240° + \sqrt{-1}.\sin 240°),$$
or $1,\ -\frac{1}{2} + \frac{\sqrt{3}}{2}\sqrt{-1},\ -\frac{1}{2} - \frac{\sqrt{3}}{2}\sqrt{-1}.$

The modulus of -1 is 1, and its argument is π. The modulus of the cube roots of -1 is 1, and their arguments are $\frac{\pi}{3}$, $\frac{\pi}{3} + \frac{2\pi}{3}$, $\frac{\pi}{3} + \frac{4\pi}{3}$, that is, $60°$, $180°$, $300°$. The roots in question, then, are represented by the points P_1, P_2, P_3, in the figure. Their values are $\frac{1}{2} + \frac{\sqrt{3}}{2}\sqrt{-1},\ -1,\ \frac{1}{2} - \frac{\sqrt{3}}{2}\sqrt{-1}.$



EXAMPLES.

(1) What are the square roots of 1 and -1? the 4th roots? the 5th roots? the 6th roots?

(2) Find the cube roots of -8; the 5th roots of 32.

(3) Show that an imaginary can have no real nth root; that a positive real has two real nth roots if n is even, one if n is odd; that a negative real has one real nth root if n is odd, none if n is even.

28. Imaginaries having equal moduli, and arguments differing only in sign, are called *conjugate imaginaries.*

$r(\cos\phi + \sqrt{-1}.\sin\phi)$, and $r[\cos(-\phi) + \sqrt{-1}.\sin(-\phi)]$, or $r(\cos\phi - \sqrt{-1}.\sin\phi)$ are *conjugate.*

They can be written $x + y\sqrt{-1}$ and $x - y\sqrt{-1}$, and we see that the points corresponding to them have the same abscissa, and ordinates which are equal with opposite signs.

EXAMPLES.

(1) Prove that *conjugate imaginaries* have a real sum and a real product.

(2) Prove, by considering in detail the substitution of $a + b\sqrt{-1}$ and $a - b\sqrt{-1}$ in turn for x in any algebraic polynomial in x with real coefficients, that if any algebraic equation with real coefficients has an imaginary root the *conjugate* of that root is also a root of the equation.

(3) Prove that if in any fraction where the numerator and denominator are rational algebraic polynomials in x, we substitute $a + b\sqrt{-1}$ and $a - b\sqrt{-1}$ in turn for x, the results are conjugate.

Transcendental Functions of Imaginaries.

29. We have adopted a definition of an *imaginary* and laid down rules to govern its use, that enable us to deal with it, in all expressions involving only algebraic operations, precisely as if it were a quantity. If we are going further, and are to subject it to *transcendental* operations, we must carefully define each function that we are going to use, and establish the rules which the function must obey.

The principal *transcendental* functions are e^x, $\log x$, and $\sin x$, and we wish to define and study these when x is replaced by an imaginary variable z.

As our conception and treatment of imaginaries have been entirely algebraic, we naturally wish to define our transcendental

functions by the aid of algebraic functions; and since we know that the transcendental functions of a *real* variable can be expressed in terms of algebraic functions only by the aid of infinite series, we are led to use such series in defining transcendental functions of an *imaginary* variable; but we must first establish a proposition concerning the convergency of a series containing imaginary terms.

30. *If the moduli of the terms of a series containing imaginary terms form a convergent series, the given series is convergent.*

Let $u_0 + u_1 + u_2 + \cdots\cdots + u_n + \cdots\cdots$ be a series containing imaginary terms.

Let

$$u_0 = R_0(\cos\Phi_0 + \sqrt{-1}.\sin\Phi_0),\ u_1 = R_1(\cos\Phi_1 + \sqrt{-1}.\sin\Phi_1),\ \&c.,$$

and suppose that the series $R_0 + R_1 + R_2 + \cdots\cdots + R_n + \cdots\cdots$ is convergent; then will the series $u_0 + u_1 + u_2 + \cdots\cdots$ be convergent.

The series $R_0 + R_1 + \cdots\cdots$ is a convergent series composed of positive terms; if then we break up the series into parts in any way, each part will have a definite sum or will approach a definite limit as the number of terms considered is increased indefinitely.

The series $u_0 + u_1 + u_2 + \cdots\cdots u_n + \cdots\cdots$ can be broken up into the two series

$$R_0\cos\Phi_0 + R_1\cos\Phi_1 + R_2\cos\Phi_2 + \cdots\cdots + R_n\cos\Phi_n + \cdots\cdots \quad (1)$$

and

$$\sqrt{-1}(R_0\sin\Phi_0 + R_1\sin\Phi_1 + R_2\sin\Phi_2 + \cdots\cdots + R_n\sin\Phi_n + \cdots\cdots). \quad (2)$$

(1) can be separated into two parts, the first made up only of positive terms, the second only of negative terms, and can therefore be regarded as the difference between two series, each consisting of positive terms. Each term in either series will be a term of the modulus series $R_0 + R_1 + R_2 + \cdots\cdots$ multiplied by a quantity less than one, and the sum of n terms of each series will therefore approach a definite limit, as n increases indefinitely. The series (1), then, which is the abscissa of the point representing the given imaginary series, has a finite sum.

In the same way it may be shown that the coefficient of $\sqrt{-1}$ in (2) has a finite sum, and this is the ordinate of the point representing the given series. The sum of n terms of the given series, then, approaches a definite limit as n is increased indefinitely, and the series is convergent.

31. We have seen (I. Art. 133 [2]) that

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\cdots \qquad [1]$$

when x is real, and that this series is convergent for all values of x.

Let us define e^z, where $z = x + y\sqrt{-1}$, by the series

$$e^z = 1 + \frac{z}{1} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots\cdots \qquad [2]$$

This series is convergent, for if $z = r(\cos\phi + \sqrt{-1}.\sin\phi)$ the series

$$1 + \frac{r}{1} + \frac{r^2}{2!} + \frac{r^3}{3!} + \frac{r^4}{4!} + \cdots\cdots$$

made up of the moduli of the terms of [2] is convergent by I. Art. 133, and therefore the value we have chosen for e^z is a determinate finite one.

Write $x + y\sqrt{-1}$ for z, and we get

$$e^{x+y\sqrt{-1}} = 1 + \frac{x+y\sqrt{-1}}{1} + \frac{(x+y\sqrt{-1})^2}{2!} + \frac{(x+y\sqrt{-1})^3}{3!} + \cdots\cdots \qquad [3]$$

The terms of this series can be expanded by the Binomial Theorem. Consider all the resulting terms containing any given power of x, say x^p; we have

$$\frac{x^p}{p!}\left(1 + \frac{y\sqrt{-1}}{1} + \frac{(y\sqrt{-1})^2}{2!} + \frac{(y\sqrt{-1})^3}{3!} + \cdots\cdots + \frac{(y\sqrt{-1})^n}{n!} + \cdots\cdots\right);$$

or, separating the real terms and the imaginary terms,

$$\frac{x^p}{p!}\left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \cdots\cdots\right)$$

$$+ \frac{x^p}{p!}\sqrt{-1}\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \cdots\cdots\right),$$

or $$\frac{x^p}{p!}(\cos y + \sqrt{-1}.\sin y), \qquad \text{by I. Art. 134.}$$

Giving p all values from 1 to ∞ we get

$$e^{x+y\sqrt{-1}} = (\cos y + \sqrt{-1}.\sin y)\left(1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\cdots\right)$$
$$= e^x(\cos y + \sqrt{-1}.\sin y), \qquad [4]$$

which, by the way, is in one of our typical imaginary forms.

If $x = 0$, in [4],

we get $$e^{y\sqrt{-1}} = \cos y + \sqrt{-1}.\sin y,$$

which suggests a new way of writing our typical imaginary; namely,

$$r(\cos\phi + \sqrt{-1}.\sin\phi) = re^{\phi\sqrt{-1}}.$$

32. We have seen that

$$e^{x+y\sqrt{-1}} = e^x e^{y\sqrt{-1}};$$

let us see if all imaginary powers of e obey the *law of indices;* that is, if the equation

$$e^u e^v = e^{u+v} \qquad [1]$$

is universally true.

Let $u = x_1 + y_1\sqrt{-1}$ and $v = x_2 + y_2\sqrt{-1}$,

then $e^u = e^{x_1 + y_1\sqrt{-1}} = e^{x_1}(\cos y_1 + \sqrt{-1}.\sin y_1)$,

$$e^v = e^{x_2 + y_2\sqrt{-1}} = e^{x_2}(\cos y_2 + \sqrt{-1}.\sin y_2),$$

$$e^u e^v = e^{x_1}e^{x_2}[\cos(y_1 + y_2) + \sqrt{-1}.\sin(y_1 + y_2)]$$
$$= e^{x_1 + x_2}[\cos(y_1 + y_2) + \sqrt{-1}.\sin(y_1 + y_2)]$$
$$= e^{x_1 + x_2 + (y_1 + y_2)\sqrt{-1}}$$
$$= e^{u+v},$$

and the *fundamental property of exponential functions holds for imaginaries as well as for reals.*

EXAMPLE.

Prove that $a^u a^v = a^{u+v}$ when u and v are imaginary.

Logarithmic Functions.

33. As a logarithm is the inverse of an exponential, we ought to be able to obtain the logarithm of an imaginary from the formula for $e^{x+y\sqrt{-1}}$. We see readily that

$$z = r(\cos\phi + \sqrt{-1}.\sin\phi) = e^{\log r + \phi\sqrt{-1}},$$

whence $$\log z = \log r + \phi\sqrt{-1};$$

or, more strictly, since

$$z = r[\cos(\phi_0 + 2n\pi) + \sqrt{-1}.\sin(\phi_0 + 2n\pi)],$$

$$\log z = \log r + (\phi_0 + 2n\pi)\sqrt{-1} \qquad [1]$$

where n is any integer.

If $z = x + y\sqrt{-1}$, $r = \sqrt{x^2 + y^2}$, and $\phi = \tan^{-1}\frac{y}{x}$;

whence $$\log z = \tfrac{1}{2}\log(x^2 + y^2) + \sqrt{-1}.\tan^{-1}\frac{y}{x}. \qquad [2]$$

Each of the expressions for $\log z$ is indeterminate, and represents an infinite number of values, differing by multiples of $2\pi\sqrt{-1}$.

This indeterminateness in the logarithm might have been expected *a priori*, for

$$e^{2\pi\sqrt{-1}} = \cos 2\pi + \sqrt{-1}.\sin 2\pi = 1, \qquad \text{by Art. 31.}$$

Hence, adding $2\pi\sqrt{-1}$ to the logarithm of any quantity will have the effect of multiplying the quantity by 1, and therefore will not change its value.

EXAMPLE.

Show that if an expression is imaginary, all its logarithms are imaginary; if it is real and positive, one logarithm is real and the rest imaginary; if it is real and negative, all are imaginary.

Trigonometric Functions.

34. If z is real,

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\cdots \qquad [1]$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\cdots \qquad [2]$$

by I. Art. 134.

If $$z = r(\cos\phi + \sqrt{-1}.\sin\phi),$$

the series of the moduli,

$$r + \frac{r^3}{3!} + \frac{r^5}{5!} + \frac{r^7}{7!} + \cdots\cdots,$$

$$1 + \frac{r^2}{2!} + \frac{r^4}{4!} + \frac{r^6}{6!} + \cdots\cdots,$$

are easily seen to be convergent; therefore if z is imaginary, the series [1] and [2] are convergent. We shall take them as definitions of the sine and cosine of an imaginary.

EXAMPLE.

From the formulas of Art. 31, and from Art. 34 [1] and [2], show that

$$e^{z\sqrt{-1}} = \cos z + \sqrt{-1}.\sin z,$$

and $$e^{-z\sqrt{-1}} = \cos z - \sqrt{-1}.\sin z,$$ for all values of z.

35. From the relations

$$e^{z\sqrt{-1}} = \cos z + \sqrt{-1}.\sin z,$$

$$e^{-z\sqrt{-1}} = \cos z - \sqrt{-1}.\sin z,$$

we get $$\cos z = \frac{e^{z\sqrt{-1}} + e^{-z\sqrt{-1}}}{2}, \qquad [1]$$

$$\sin z = \frac{e^{z\sqrt{-1}} - e^{-z\sqrt{-1}}}{2\sqrt{-1}}, \qquad [2]$$

for all values of z.

Let $$z = x + y\sqrt{-1}.$$

$$\cos(x+y\sqrt{-1}) = \frac{e^{x\sqrt{-1}-y} + e^{-x\sqrt{-1}+y}}{2}$$

$$= \frac{(\cos x + \sqrt{-1}.\sin x)e^{-y} + (\cos x - \sqrt{-1}.\sin x)e^{y}}{2},$$

by Art. 34, Ex.,

$$= \cos x\frac{e^y + e^{-y}}{2} - \sqrt{-1}.\sin x\frac{e^y - e^{-y}}{2}. \qquad [3]$$

In the same way it may be shown that

$$\sin(x+y\sqrt{-1}) = \frac{(\cos x + \sqrt{-1}.\sin x)e^{-y} - (\cos x - \sqrt{-1}.\sin x)e^{y}}{2\sqrt{-1}}$$

$$= \sin x\frac{e^y + e^{-y}}{2} + \sqrt{-1}.\cos x\frac{e^y - e^{-y}}{2}. \qquad [4]$$

If z is real in [1] and [2], we have

$$\cos x = \frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2},$$

$$\sin x = -\frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2}\sqrt{-1}.$$

If $z = y\sqrt{-1}$, and is a pure imaginary,

$$\cos y\sqrt{-1} = \frac{e^y + e^{-y}}{2}, \qquad [5]$$

$$\sin y\sqrt{-1} = \frac{e^y - e^{-y}}{2}\sqrt{-1}; \qquad [6]$$

whence we see that the cosine of a pure imaginary is real, while its sine is imaginary.

By the aid of [5] and [6], [3] and [4] can be written:

$$\cos(x + y\sqrt{-1}) = \cos x\cos y\sqrt{-1} - \sin x\sin y\sqrt{-1}, \quad [7]$$

$$\sin(x + y\sqrt{-1}) = \sin x\cos y\sqrt{-1} + \cos x\sin y\sqrt{-1}. \quad [8]$$

EXAMPLES.

(1) From [1] and [2] show that $\sin^2 z + \cos^2 z = 1$.

(2) Prove that

$$\cos(u+v) = \cos u \cos v - \sin u \sin v,$$

$$\sin(u+v) = \sin u \cos v + \cos u \sin v,$$

where u and v are imaginary.

The relations to be proved in examples (1) and (2) are the fundamental formulas of Trigonometry, and they enable us to use trigonometric functions of imaginaries precisely as we use trigonometric functions of reals.

Differentiation of Functions of Imaginaries.

36. A function of an imaginary variable,

$$z = x + y\sqrt{-1},$$

is, strictly speaking, a function of two independent variables, x and y; for we can change z by changing either x or y, or both x and y. Its differential will usually contain dx and dy, and not necessarily dz; and if we divide its differential by dz to get its derivative with respect to z, the result will generally contain $\frac{dy}{dx}$, which will be wholly indeterminate, since x and y are entirely independent in the expression $x + y\sqrt{-1}$. It may happen, however, in the case of some simple functions, that dz will appear as a factor in the differential of the function, which in that case will have a single derivative.

37. *In differentiating, the* $\sqrt{-1}$ *may be treated like a constant;* for the operation of finding the differential of a function is an algebraic operation, and in all algebraic operations $\sqrt{-1}$ obeys the same laws as any constant.

EXAMPLE.

Prove that $d(x^2\sqrt{-1}) = 2x\sqrt{-1}.\,dx;$

and that $d\sqrt{-1}.\sin x = \sqrt{-1}.\cos x.dx \ldots\ldots$

We have, by the aid of this principle,

if $$z = x + y\sqrt{-1},$$
$$dz = dx + \sqrt{-1}.\,dy; \qquad [1]$$

if $$z = r(\cos\phi + \sqrt{-1}.\sin\phi),$$
$$dz = dr(\cos\phi + \sqrt{-1}.\sin\phi) + rd\phi(-\sin\phi + \sqrt{-1}.\cos\phi)$$
$$= (dr + r\sqrt{-1}.d\phi)(\cos\phi + \sqrt{-1}.\sin\phi). \qquad [2]$$

38. Let us now consider the differentiation of z^m, e^z, $\log z$, $\sin z$, and $\cos z$.

Let $$z = r(\cos\phi + \sqrt{-1}.\sin\phi),$$

then
$$z^m = r^m(\cos m\phi + \sqrt{-1}.\sin m\phi), \qquad \text{by Art. 24 [1]};$$
$$dz^m = mr^{m-1}dr(\cos m\phi + \sqrt{-1}.\sin m\phi) + mr^m d\phi(-\sin m\phi + \sqrt{-1}.\cos m\phi),$$
$$dz^m = mr^{m-1}[\cos(m-1)\phi + \sqrt{-1}.\sin(m-1)\phi](\cos\phi + \sqrt{-1}.\sin\phi)dr$$
$$+ mr^m[\cos(m-1)\phi + \sqrt{-1}.\sin(m-1)\phi](\cos\phi + \sqrt{-1}.\sin\phi)\sqrt{-1}.d\phi,$$
$$dz^m = mr^{m-1}[\cos(m-1)\phi + \sqrt{-1}.\sin(m-1)\phi](dr + r\sqrt{-1}.d\phi)(\cos\phi + \sqrt{-1}.\sin\phi),$$
$$dz^m = mz^{m-1}dz, \qquad [1] \quad \text{by Art. 37 [2]},$$
$$\frac{dz^m}{dz} = mz^{m-1}, \qquad [2]$$
and a power of an imaginary variable has a single derivative.

EXAMPLE.

Show that [1] and [2] hold for all powers of z.

39. If $z = x + y\sqrt{-1}$,

$$e^z = e^x(\cos y + \sqrt{-1}.\sin y), \qquad \text{by Art. 31 [1]},$$

$$de^z = e^x dx(\cos y + \sqrt{-1}.\sin y) + e^x(-\sin y + \sqrt{-1}.\cos y)\,dy,$$

$$de^z = e^z(\cos y + \sqrt{-1}.\sin y)\,(dx + \sqrt{-1}.dy),$$

$$de^z = e^z dz, \qquad [1]$$

$$\frac{de^z}{dz} = e^z. \qquad [2]$$

EXAMPLE.

Show that $da^z = a^z \log a.dz$.

40. If $z = r(\cos\phi + \sqrt{-1}.\sin\phi)$,

$$\log z = \log r + \phi\sqrt{-1}, \qquad \text{by Art. 33},$$

$$d\log z = \frac{dr}{r} + \sqrt{-1}.\,d\phi = \frac{dr + r\sqrt{-1}.d\phi}{r},$$

$$d\log z = \frac{(dr + r\sqrt{-1}.d\phi)(\cos\phi + \sqrt{-1}.\sin\phi)}{r(\cos\phi + \sqrt{-1}.\sin\phi)},$$

$$d\log z = \frac{dz}{z}, \qquad [1]$$

$$\frac{d\log z}{dz} = \frac{1}{z}. \qquad [2]$$

41. $$\sin z = \frac{e^{z\sqrt{-1}} - e^{-z\sqrt{-1}}}{2\sqrt{-1}}, \qquad \text{by Art. 35 [2]},$$

$$d\sin z = \frac{e^{z\sqrt{-1}} + e^{-z\sqrt{-1}}}{2\sqrt{-1}}\sqrt{-1}.\,dz$$

$$= \frac{e^{z\sqrt{-1}} + e^{-z\sqrt{-1}}}{2}\,dz, \qquad \text{by Art. 35 [1]},$$

$$d\sin z = \cos z.dz. \qquad [1]$$

$$\cos z = \frac{e^{z\sqrt{-1}} + e^{-z\sqrt{-1}}}{2},$$

$$d\cos z = \frac{e^{z\sqrt{-1}} - e^{-z\sqrt{-1}}}{2}\sqrt{-1}.dz = -\frac{e^{z\sqrt{-1}} - e^{-z\sqrt{-1}}}{2\sqrt{-1}}dz,$$

$$d\cos z = -\sin z.dz. \qquad [2]$$

42. We see, then, that we get the same formulas for the differentiation of simple functions of imaginaries as for the differentiation of the corresponding functions of reals. It follows that our formulas for direct integration (I. Art. 74) hold when x is imaginary.

Hyperbolic Functions.

43. We have (Art. 35 [5] and [6])

$$\cos x\sqrt{-1} = \frac{e^x + e^{-x}}{2},$$

and
$$\sin x\sqrt{-1} = \frac{e^x - e^{-x}}{2}\sqrt{-1},$$

where x is real. $\frac{e^x + e^{-x}}{2}$ is called the hyperbolic cosine of x, and is written $\mathrm{Ch}\,x$; and $\frac{e^x - e^{-x}}{2}$ is called the hyperbolic sine of x, and is written $\mathrm{Sh}\,x$:

$$\mathrm{Sh}\,x = \frac{e^x - e^{-x}}{2} = -\sqrt{-1}.\sin x\sqrt{-1}, \qquad [1]$$

$$\mathrm{Ch}\,x = \frac{e^x + e^{-x}}{2} = \cos x\sqrt{-1}. \qquad [2]$$

The hyperbolic tangent is defined as the ratio of Sh to Ch; and the hyperbolic cotangent, secant, and cosecant are the reciprocals of the Th, Ch, and Sh respectively.

These functions, which are real when x is real, resemble in their properties the ordinary trigonometric functions.

44. For example,

$$\text{Ch}^2 x - \text{Sh}^2 x = 1; \qquad [1]$$

for
$$\text{Ch}^2 x = \frac{e^{2x} + 2 + e^{-2x}}{4},$$

and
$$\text{Sh}^2 x = \frac{e^{2x} - 2 + e^{-2x}}{4}.$$

EXAMPLES.

(1) Prove that $1 - \text{Th}^2 x = \text{Sch}^2 x.$

(2) Prove that $1 - \text{Cth}^2 x = -\text{Csch}^2 x.$

(3) Prove that $\text{Sh}(x+y) = \text{Sh}\,x\,\text{Ch}\,y + \text{Ch}\,x\,\text{Sh}\,y.$

(4) Prove that $\text{Ch}(x+y) = \text{Ch}\,x\,\text{Ch}\,y + \text{Sh}\,x\,\text{Sh}\,y.$

45.
$$d\,\text{Sh}\,x = d\,\frac{e^x - e^{-x}}{2} = \frac{e^x + e^{-x}}{2}\,dx,$$
$$d\,\text{Sh}\,x = \text{Ch}\,x.dx.$$

EXAMPLES.

(1) Prove
$$d\,\text{Ch}\,x = \text{Sh}\,x.dx.$$
$$d\,\text{Th}\,x = \text{Sch}^2 x.dx.$$
$$d\,\text{Cth}\,x = -\text{Csch}^2 x.dx.$$
$$d\,\text{Sch}\,x = -\text{Sch}\,x\,\text{Th}\,x.dx.$$
$$d\,\text{Csch}\,x = -\text{Csch}\,x\,\text{Cth}\,x.dx.$$

46. We can deal with anti-hyperbolic functions just as with anti-trigonometric functions.

To find $d\,\text{Sh}^{-1}x$.

Let
$$u = \text{Sh}^{-1}x,$$
then
$$x = \text{Sh}\,u,$$
$$dx = \text{Ch}\,u.du,$$

$$du = \frac{dx}{\text{Ch}\, u},$$

$$\text{Ch}\, u = \sqrt{1 + \text{Sh}^2 u}, \qquad \text{by Art. 4 [1]},$$

$$\text{Ch}\, u = \sqrt{1 + x^2},$$

$$d\,\text{Sh}^{-1} x = \frac{dx}{\sqrt{1 + x^2}}. \qquad [1]$$

EXAMPLES.

Prove the formulas

$$d\,\text{Ch}^{-1} x = \frac{dx}{\sqrt{x^2 - 1}}.$$

$$d\,\text{Th}^{-1} x = \frac{dx}{1 - x^2}.$$

$$d\,\text{Sch}^{-1} x = -\frac{dx}{x\sqrt{1 - x^2}}.$$

$$d\,\text{Csch}^{-1} x = -\frac{dx}{x\sqrt{x^2 + 1}}.$$

47. The anti-hyperbolic functions are easily expressed as logarithms.

Let
$$u = \text{Sh}^{-1} x,$$

then
$$x = \text{Sh}\, u = \frac{e^u - e^{-u}}{2},$$

$$2x = e^u - \frac{1}{e^u},$$

$$2xe^u = e^{2u} - 1,$$

$$e^{2u} - 2xe^u = 1,$$

$$e^{2u} - 2xe^u + x^2 = 1 + x^2,$$

$$e^u - x = \pm\sqrt{1 + x^2},$$

$$e^u = x \pm \sqrt{1 + x^2};$$

as e^u is necessarily positive, we may reject the negative value in the second member as impossible, and we have

$$e^u = x + \sqrt{1+x^2},$$

$$u = \log(x + \sqrt{1+x^2}),$$

or

$$\text{Sh}^{-1}x = \log(x + \sqrt{1+x^2}). \qquad [1]$$

EXAMPLES.

Prove the formulas

$$\text{Ch}^{-1}x = \log(x + \sqrt{x^2-1}).$$

$$\text{Th}^{-1}x = \tfrac{1}{2}\log\frac{1+x}{1-x}.$$

$$\text{Sch}^{-1}x = \log\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1}\right).$$

$$\text{Csch}^{-1}x = \log\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1}\right).$$

48. The principal advantage arising from the use of hyperbolic functions is that they bring to light some curious analogies between the integrals of certain irrational functions.

From I. Art. 71 we obtain the formulas for direct integration.

$$\int\frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x. \qquad [1]$$

$$\int\frac{dx}{1+x^2} = \tan^{-1}x. \qquad [2]$$

$$\int\frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}x. \qquad [3]$$

From Art. 46 we obtain the allied formulas:

$$\int\frac{dx}{\sqrt{1+x^2}} = \text{Sh}^{-1}x = \log(x + \sqrt{1+x^2}). \qquad [4]$$

$$\int\frac{dx}{\sqrt{x^2-1}} = \text{Ch}^{-1}x = \log(x + \sqrt{x^2-1}). \qquad [5]$$

$$\int \frac{dx}{1-x^2} \quad = \mathrm{Th}^{-1} x \quad = \tfrac{1}{2} \log \frac{1+x}{1-x}. \qquad [6]$$

$$-\int \frac{dx}{x\sqrt{1-x^2}} = \mathrm{Sch}^{-1} x \ = \log\left(\frac{1}{x} + \sqrt{\frac{1}{x^2}-1}\right). \qquad [7]$$

$$-\int \frac{dx}{x\sqrt{x^2+1}} = \mathrm{Csch}^{-1} x = \log\left(\frac{1}{x} + \sqrt{\frac{1}{x^2}+1}\right). \qquad [8]$$

CHAPTER III.

GENERAL METHODS OF INTEGRATING.

49. We have defined the *integral* of any function of a single variable as *the function which has the given function for its derivative* (I. Art. 53); we have defined a *definite integral* as the *limit of the sum of a set of differentials;* and we have shown that a definite integral is the *difference between two values of an ordinary integral* (I. Art. 183).

Now that we have adopted the differential notation in place of the derivative notation, it is better to regard an integral as the inverse of a *differential* instead of as the inverse of a *derivative.* Hence the integral of $fx.dx$ will be the function whose differential is $fx.dx$; and we shall indicate it by $\int fx.dx$. In our old notation we should have indicated precisely the same function by $\int_x fx$; for if the derivative of a function is fx we know that its differential is $fx.dx$.

50. If fx is any function whatever of x, fx.dx *has an integral.* For if we construct the curve whose equation is $y = fx$, we know that the area included by the curve, the axis of X, any fixed ordinate, and the ordinate corresponding to the variable x, has for its differential ydx, or, in other words, $fx.dx$ (I. Art. 51). Such an area always exists, and it is a determinate function of x, except that, as the position of the initial ordinate is wholly arbitrary, the expression for the area will contain an arbitrary constant. Thus, if Fx is the area in question for some one position of the initial ordinate, we shall have

$$\int fx.dx = Fx + C,$$

where C is an arbitrary constant.

Moreover, $Fx + C$ is a complete expression for $\int fx.dx$; for if two functions of x have the same differential, they have the same derivative with respect to x, and therefore they change at the same rate when x changes (I. Art. 38); they can differ, then, at any instant only by the difference between their initial values, which is some constant.

Hence we see that *every expression of the form* fx.dx *has an integral, and, except for the presence of an arbitrary constant, but one integral.*

51. We have shown in I. Art. 183 that a *definite integral* is the difference between two values of an ordinary integral, and therefore contains no constant. Thus, if $Fx + C$ is the integral of $fx.dx$,

$$\int_a^b fx.dx = Fb - Fa.$$

In the same way we shall have

$$\int_a^b fz.dz = Fb - Fa;$$

and we see that a *definite integral is a function of the values between which the sum is taken* and not of the variable with respect to which we integrate.

Since

$$\int_b^a fx.dx = Fa - Fb,$$

$$\int_b^a fx.dx = -\int_a^b fx.dx.$$

EXAMPLE.

Show that $\int_a^c fx.dx + \int_c^b fx.dx = \int_a^b fx.dx.$

52. In what we have said concerning definite integrals we have tacitly assumed that the integral is a *continuous function* between the values between which the sum in question is taken. If it is not, we cannot regard the whole increment of Fx as equal

to the limit of the sum of the partial infinitesimal increments, and the reasoning of I. Art. 183 ceases to be valid.

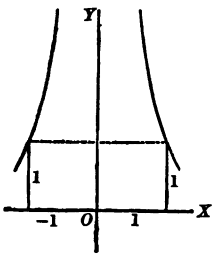
Take, for example, $\int_{-1}^{1}\frac{dx}{x^2}$.

$$\int\frac{dx}{x^2}=\int x^{-2}dx=\frac{x^{-1}}{-1}=-\frac{1}{x},\quad \text{by I. Art. 55 (7);}$$

and apparently

$$\int_{-1}^{1}\frac{dx}{x^2}=\left(-\frac{1}{x}\right)_{x=1}-\left(-\frac{1}{x}\right)_{x=-1}=-2.$$

But $\int_{-1}^{1}\frac{dx}{x^2}$ ought to be the area between the curve $y=\frac{1}{x^2}$, the axis of x, and the ordinates corresponding to $x=1$ and $x=-1$, which evidently is not -2; and we see that the function $\frac{1}{x^2}$ is discontinuous between the values $x=-1$ and $x=1$.



The area in question which the definite integral should represent is easily seen to be infinite, for

$$\int_{-1}^{-\epsilon}\frac{dx}{x^2}=\frac{1}{\epsilon}-1,\text{ and }\int_{\epsilon}^{1}\frac{dx}{x^2}=\frac{1}{\epsilon}-1,$$

and each of these expressions increases without limit as ϵ approaches zero.

53. Since a definite integral is the difference between two values of an indefinite integral, what we have to find first in any problem is the indefinite integral. This may be found by inspection if the function to be integrated comes under any of the forms we have already obtained by differentiation, and we are then said to integrate directly. Direct integration has been illustrated, and the most important of the forms which can be integrated directly have been given in I. Chapter V. For the sake of convenience we rewrite these forms, using the differential notation, and adding one or two new forms from our sections on hyperbolic functions.

$$\int x^n dx = \frac{x^{n+1}}{n+1}.$$

$$\int \frac{dx}{x} = \log x.$$

$$\int a^x dx = \frac{a^x}{\log a}.$$

$$\int e^x dx = e^x.$$

$$\int \sin x . dx = -\cos x.$$

$$\int \cos x . dx = \sin x.$$

$$\int \tan x . dx = -\log \cos x.$$

$$\int \operatorname{ctn} x . dx = \log \sin x.$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x.$$

$$\int \frac{dx}{\sqrt{1+x^2}} = \operatorname{Sh}^{-1} x = \log(x + \sqrt{1+x^2}).$$

$$\int \frac{dx}{\sqrt{x^2-1}} = \operatorname{Ch}^{-1} x = \log(x + \sqrt{x^2-1}).$$

$$\int \frac{dx}{1+x^2} = \tan^{-1} x.$$

$$\int \frac{dx}{1-x^2} = \operatorname{Th}^{-1} x = \tfrac{1}{2} \log \frac{1+x}{1-x}.$$

$$\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x.$$

$$\int \frac{dx}{x\sqrt{1-x^2}} = -\operatorname{Sch}^{-1} x = -\log\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1}\right).$$

$$\int \frac{dx}{x\sqrt{x^2+1}} = -\operatorname{Csch}^{-1} x = -\log\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1}\right).$$

$$\int \frac{dx}{\sqrt{2x-x^2}} = \operatorname{vers}^{-1} x.$$

54. We took up in I. Chap. V. the principal devices used in preparing a function for integration when it cannot be integrated directly.

The first of these methods, that of *integration by substitution*, is simplified by the use of the differential notation, because the formula for *change of variable* (I. Art. 75 [1]),

$$\int_x u = \int_y u D_y x \quad \text{becoming} \quad \int u dx = \int u \frac{dx}{dy} dy,$$

reduces to an identity and is no longer needed, and all that is required is a simple substitution.

(*a*) For example, let us find $\int \frac{dx}{x} \sqrt{1 + \log x}$.

Let $$1 + \log x = z,$$

then $$\frac{dx}{x} = dz;$$

and $$\int \frac{dx}{x} \sqrt{1 + \log x} = \int z^{\frac{1}{2}} dz = \tfrac{2}{3} z^{\frac{3}{2}} = \tfrac{2}{3}(1 + \log x)^{\frac{3}{2}}.$$

(*b*) Required $\int \frac{dx}{e^x + e^{-x}}$.

Let $$e^x = y,$$

then $$e^x dx = dy,$$

$$\frac{dx}{e^x + e^{-x}} = \frac{e^x dx}{e^{2x} + 1} = \frac{dy}{y^2 + 1};$$

and $$\int \frac{dx}{e^x + e^{-x}} = \int \frac{dy}{1 + y^2} = \tan^{-1} y = \tan^{-1} e^x.$$

(*c*) Required $\int \sec x . dx$.

$$\sec x = \frac{1}{\cos x} = \frac{\cos x}{\cos^2 x}.$$

Let $$z = \sin x;$$

then $$dz = \cos x . dx.$$

$$\cos^2 x = 1 - z^2,$$

$$\int\frac{\cos x.dx}{\cos^2 x}=\int\frac{dz}{1-z^2}=\tfrac{1}{2}\log\frac{1+z}{1-z},\qquad \text{by Art. 53,}$$

$$\int\sec x.dx=\tfrac{1}{2}\log\frac{1+\sin x}{1-\sin x}.$$

EXAMPLES.

Prove that (1) $\int\csc x.dx=\tfrac{1}{2}\log\frac{1-\cos x}{1+\cos x}=\log\tan\frac{x}{2}.$

(2) $\int\frac{x^2dx}{\sqrt{1-x^2}}=-\tfrac{1}{2}\cos^{-1}x-\frac{x\sqrt{1-x^2}}{2}.$

Suggestion: Let $x=\cos z$.

55. The formula for *integration by parts* (I. Art. 79 [1]) becomes

$$\int udv=uv-\int vdu, \tag{1}$$

when we use the differential notation. It is used as in I. Chap. V.

(*a*) For example, let us find $\int x^n\log x.dx$.

Let $u=\log x,$

and $dv=x^n dx;$

then $du=\frac{dx}{x},$

and $v=\frac{x^{n+1}}{n+1},$

$$\int x^n\log x.dx=\frac{x^{n+1}}{n+1}\log x-\int\frac{x^n}{n+1}dx=\frac{x^{n+1}}{n+1}\left(\log x-\frac{1}{n+1}\right).$$

(*b*) Required $\int x\sin^{-1}x.dx.$

Let $u=\sin^{-1}x,$

and $dv=xdx;$

then $du=\frac{dx}{\sqrt{1-x^2}},$

and $$v = \frac{x^2}{2},$$

$$\int x\sin^{-1}x.dx = \frac{x^2}{2}\sin^{-1}x - \tfrac{1}{2}\int\frac{x^2\,dx}{\sqrt{1-x^2}},$$

$$\int x\sin^{-1}x.dx = \frac{x^2}{2}\sin^{-1}x + \tfrac{1}{4}(\cos^{-1}x + x\sqrt{1-x^2}).$$ (?)

(c) Required $\int\frac{xe^x\,dx}{(1+x)^2}$.

Let $$u = xe^x,$$

and $$dv = \frac{dx}{(1+x)^2};$$

then $$du = (xe^x + e^x)dx = e^x(1+x)\,dx,$$

and $$v = -\frac{1}{1+x},$$

$$\int\frac{xe^x\,dx}{(1+x)^2} = -\frac{xe^x}{1+x} + \int e^x\,dx = -\frac{xe^x}{1+x} + e^x = \frac{e^x}{1+x^2}.$$

EXAMPLES.

(1) $\displaystyle\int\frac{dx}{\sqrt{1-3x-x^2}} = \sin^{-1}\frac{3+2x}{\sqrt{13}}.$

(2) $\displaystyle\int x\tan^{-1}x.dx = \frac{1+x^2}{2}\tan^{-1}x - \tfrac{1}{2}x.$

(3) $\displaystyle\int\frac{xdx}{(1-x)^3} = -\frac{1}{1-x} + \frac{1}{2(1-x)^2}.$

(4) $\displaystyle\int\frac{xdx}{\sqrt{2ax-x^2}} = -\sqrt{2ax-x^2} + a\,\text{vers}^{-1}\frac{x}{a}.$

(5) $\displaystyle\int\sqrt{2ax-x^2}.\,dx = \frac{x-a}{2}\sqrt{2ax-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x-a}{a}.$

Suggestion: Throw $2ax - x^2$ into the form $a^2 - (x-a)^2$.

(6) $\displaystyle\int\frac{1+\cos x}{x+\sin x}\,dx = \log(x+\sin x).$

(7) $\int \frac{x + \sin x}{1 + \cos x} dx = x \tan \frac{x}{2}.$

Suggestion: Introduce $\frac{x}{2}$ in place of x.

(8) $\int \frac{dx}{x(\log x)^n} = -\frac{1}{(n-1)(\log x)^{n-1}}.$

(9) $\int \frac{\log(\log x)}{x} dx = \log x [\log(\log x) - 1].$

(10) $\int \frac{\sin^{-1} x . dx}{(1 - x^2)^{\frac{3}{2}}} = z \tan z + \log \cos z,$ where $z = \sin^{-1} x.$

(11) $\int \frac{dx}{\sin x + \cos x} = \frac{1}{\sqrt{2}} \log \tan \left(\frac{x}{2} + \frac{\pi}{8}\right).$

CHAPTER IV.

RATIONAL FRACTIONS.

56. We shall now attempt to consider systematically the methods of integrating various functions; and to this end we shall begin with *rational algebraic expressions*. *Any rational algebraic polynomial can be integrated immediately* by the aid of the formula

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}.$$

Take next a *rational fraction*, that is, a fraction whose numerator and denominator are rational algebraic polynomials. A rational fraction is *proper* if its numerator is of lower degree than its denominator; *improper* if the degree of the numerator is equal to or greater than the degree of the denominator. Since an improper fraction can always be reduced to a polynomial plus a proper fraction, by actually dividing the numerator by the denominator, we need only consider the treatment of proper fractions.

57. *Every proper rational fraction can be reduced to the sum of a set of simpler fractions each of which has a constant for a numerator and some power of a binomial for its denominator;* that is, a set of fractions any one of which is of the form $\frac{A}{(x-a)^m}$.

Let our given fraction be $\frac{fx}{Fx}$.

If a, b, c, &c., are the roots of the equation,

$$Fx = 0, \tag{1}$$

we have, from the Theory of Equations,

$$Fx = A(x-a)(x-b)(x-c)\cdots\cdots \tag{2}$$

The equation (1) may have some equal roots, and then some of the factors in (2) will be repeated. Suppose a occurs p times as a root of (1), b occurs q times, c occurs r times, &c.,

then $$Fx = A(x-a)^p(x-b)^q(x-c)^r \cdots\cdots \qquad (3)$$

Call $$A(x-b)^q(x-c)^r \cdots\cdots = \phi x;$$

then $$Fx = (x-a)^p \phi x,$$

and $$\frac{fx}{Fx} = \frac{fx}{(x-a)^p\phi x} = \frac{fx - \frac{fa}{\phi a}\phi x}{(x-a)^p\phi x} + \frac{\frac{fa}{\phi a}\phi x}{(x-a)^p\phi x}$$

$$= \frac{\frac{fa}{\phi a}}{(x-a)^p} + \frac{fx - \frac{fa}{\phi a}\phi x}{(x-a)^p\phi x}.$$

$\dfrac{fx - \frac{fa}{\phi a}\phi x}{(x-a)^p\phi x}$ is a new proper fraction, but it can be reduced to a simpler form by dividing numerator and denominator by $x-a$, which is an exact divisor of the numerator because a is a root of the equation

$$fx - \frac{fa}{\phi a}\phi x = 0.$$

If we represent by $f_1 x$ the quotient arising from the division of $fx - \frac{fa}{\phi a}\phi x$ by $x-a$, we shall have

$$\frac{fx}{Fx} = \frac{\frac{fa}{\phi a}}{(x-a)^p} + \frac{f_1 x}{(x-a)^{p-1}\phi x},$$

where $\dfrac{f_1 x}{(x-a)^{p-1}\phi x}$ is a proper fraction, and may be treated precisely as we have treated the original fraction.

Hence $$\frac{f_1 x}{(x-a)^{p-1}\phi x} = \frac{\frac{f_1 a}{\phi a}}{(x-a)^{p-1}} + \frac{f_2 x}{(x-a)^{p-2}\phi x}.$$

By continuing this process we shall get

$$\frac{fx}{Fx} = \frac{\frac{fa}{\phi a}}{(x-a)^p} + \frac{\frac{f_1 a}{\phi a}}{(x-a)^{p-1}} + \frac{\frac{f_2 a}{\phi a}}{(x-a)^{p-2}} + \cdots\cdots + \frac{\frac{f_{p-1} a}{\phi a}}{x-a} + \frac{f_p x}{\phi x}.$$

In the same way $\frac{f_p x}{\phi x}$ can be broken up into a set of fractions having $(x-b)^q$, $(x-b)^{q-1}$, &c., for denominators, plus a fraction which can be broken up into fractions having $(x-c)^r$, $(x-c)^{r-1}$ ....., &c., for denominators; and we shall have, in the end,

$$\frac{fx}{Fx}=\frac{A_1}{(x-a)^p}+\frac{A_2}{(x-a)^{p-1}}+\dots\dots+\frac{A_p}{x-a}+\frac{B_1}{(x-b)^q}$$
$$+\frac{B_2}{(x-b)^{q-1}}+\dots\dots+\frac{B_q}{x-b}+\dots\dots+K, \qquad [1]$$

where K is the quotient obtained when we divide out the last factor of the denominator, and is consequently a constant. More than this, K must be zero, for as (1) is identically true, it must be true when $x=\infty$; but when $x=\infty$, $\frac{fx}{Fx}$ becomes zero, because its denominator is of higher degree than its numerator, and each of the fractions in the second member also becomes zero; whence $K=0$.

58. Since we now know the form into which any given rational fraction can be thrown, we can determine the numerators by the aid of known properties of an identical equation.

Let it be required to break up $\frac{3x-1}{(x-1)^2(x+1)}$ into simpler fractions.

By Art. 57,

$$\frac{3x-1}{(x-1)^2(x+1)}=\frac{A}{(x-1)^2}+\frac{B}{x-1}+\frac{C}{x+1},$$

and we wish to determine A, B, and C. Clearing of fractions, we have

$$3x-1=A(x+1)+B(x-1)(x+1)+C(x-1)^2. \qquad (1)$$

As this equation is identically true, the coefficients of like powers of x in the two members must be equal; and we have

$$B+C=0,$$
$$A-2C=3,$$
$$A-B+C=-1;$$

whence we find
$$A = 1,$$
$$B = 1,$$
$$C = -1;$$

and
$$\frac{3x-1}{(x-1)^2(x+1)} = \frac{1}{(x-1)^2} + \frac{1}{x-1} - \frac{1}{x+1}. \qquad (2)$$

The labor of determining the required constants can often be lessened by simple algebraic devices.

For example; since the identical equation we start with is true for all values of x, we have a right to substitute for x values that will make terms of the equation disappear. Take equation [1]:

$$3x-1 = A(x+1) + B(x+1)(x-1) + C(x-1)^2. \qquad [1]$$

Let $x = 1$, $\qquad 2 = 2A,$
$$A = 1,$$
then $\qquad 2x - 2 = B(x+1)(x-1) + C(x-1)^2;$

divide by $x-1$, $\qquad 2 = B(x+1) + C(x-1).$

Let $x = 1$, $\qquad 2 = 2B,$
$$B = 1,$$
then $\qquad -x + 1 = C(x-1),$
$$C = -1.$$

EXAMPLES.

(1) Show that when we equate the coefficients of the same powers of x on the two sides of our identical equation, we shall always have equations enough to determine all our required numerators.

(2) Break up $\dfrac{9x^2+9x-128}{(x-3)^2(x+1)}$ into simpler fractions.

59. The partial fractions corresponding to any given factor of the denominator can be determined directly.

Let us suppose that the factor in question is of the first degree and occurs but once; represent it by $x-a$.

$$\frac{fx}{Fx}=\frac{A}{x-a}+\frac{f_1x}{\phi x}, \tag{1}$$

by Art. 57, where

$$\phi x=\frac{Fx}{x-a},$$

so that $$Fx=(x-a)\phi x.$$

Clear (1) of fractions.

$$fx=A\phi x+(x-a)f_1x. \tag{2}$$

As (1) is an identical equation, (2) will be true for any value of x. Let $x=a$,

$$fa=A\phi a,$$

$$A=\frac{fa}{\phi a}, \tag{3}$$

a result agreeing with Art. 57.

Hence, *to find the numerator of the fraction corresponding to a factor* (x − a) *of the first degree, we have merely to strike out from the denominator of our original fraction the factor in question, and then substitute* a *for* x *in the result.*

If the factor of the denominator is of the nth degree, there are n partial fractions corresponding to it. Let $(x-a)^n$ be the factor in question.

$$\frac{fx}{Fx}=\frac{A_1}{(x-a)^n}+\frac{A_2}{(x-a)^{n-1}}+\frac{A_3}{(x-a)^{n-2}}+\cdots\cdots+\frac{A_n}{x-a}+\frac{\psi x}{\phi x}, \tag{4}$$

where $$Fx=(x-a)^n\phi x.$$

Multiply (4) by $(x-a)^n$, and represent $(x-a)^n\dfrac{fx}{Fx}$ by Φx.

$$\Phi x=A_1+A_2(x-a)+A_3(x-a)^2+\cdots\cdots+A_n(x-a)^{n-1}$$
$$+\frac{\psi x}{\phi x}(x-a)^n.$$

Differentiate successively both members of this identity, and put $x=a$ after differentiation, and we get

$$A_1 = \Phi a,$$

$$A_2 = \Phi' a,$$

$$A_3 = \frac{1}{2\,!}\Phi'' a,$$

$$A_4 = \frac{1}{3\,!}\Phi''' a,$$

$$\ldots\ldots,$$

$$A_n = \frac{1}{(n-1)\,!}\Phi^{(n-1)} a.$$

Although these results form a complete solution of the problem, and one exceedingly neat in theory, the labor of getting the successive derivatives of Φx is so great that it is usually easier in practice to use the methods of Art. 58 when we have to deal with factors of higher degree than the first. So far as the fractions corresponding to factors of the first degree are concerned, the method of this article can be profitably combined with that of Art. 58.

60. As an example where the method of the last article applies well, consider

$$\frac{3x-1}{x(x-2)(x+1)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+1},$$

$$A = \left[\frac{3x-1}{(x-2)(x+1)}\right]_{x=0} = \frac{1}{2},$$

$$B = \left[\frac{3x-1}{x(x+1)}\right]_{x=2} = \frac{5}{6},$$

$$C = \left[\frac{3x-1}{x(x-2)}\right]_{x=-1} = -\frac{4}{3},$$

$$\frac{3x-1}{x(x-2)(x+1)} = \frac{1}{2}\cdot\frac{1}{x} + \frac{5}{6}\cdot\frac{1}{x-2} - \frac{4}{3}\cdot\frac{1}{x+1}. \qquad [1]$$

Again, consider

$$\frac{1}{1+x^2}=\frac{1}{(x+\sqrt{-1})(x-\sqrt{-1})}=\frac{A}{x+\sqrt{-1}}+\frac{B}{x-\sqrt{-1}},$$

$$A=\left[\frac{1}{x-\sqrt{-1}}\right]_{x=-\sqrt{-1}}=-\frac{1}{2\sqrt{-1}}=\frac{\sqrt{-1}}{2},$$

$$B=\left[\frac{1}{x+\sqrt{-1}}\right]_{x=\sqrt{-1}}=\frac{1}{2\sqrt{-1}}=-\frac{\sqrt{-1}}{2},$$

$$\frac{1}{1+x^2}=\frac{\sqrt{-1}}{2}\cdot\frac{1}{x+\sqrt{-1}}-\frac{\sqrt{-1}}{2}\cdot\frac{1}{x-\sqrt{-1}}. \qquad [2]$$

61. Let us now consider a more difficult example, where it is worth while to combine our methods.

To break up $\dfrac{x^2+1}{(x-1)^4(x^3+1)}$.

$$x^3+1=(x+1)(x^2-x+1)$$

$$=(x+1)(x-\tfrac{1}{2}-\tfrac{1}{2}\sqrt{-3})(x-\tfrac{1}{2}+\tfrac{1}{2}\sqrt{-3}),$$

$$\frac{x^2+1}{(x-1)^4(x^3+1)}=\frac{x^2+1}{(x-1)^4(x+1)(x^2-x+1)}=\frac{A_1}{(x-1)^4}$$

$$+\frac{A_2}{(x-1)^3}+\frac{A_3}{(x-1)^2}+\frac{A_4}{x-1}+\frac{B}{x+1}+\frac{C}{x-\frac{1}{2}-\frac{1}{2}\sqrt{-3}}$$

$$+\frac{D}{x-\frac{1}{2}+\frac{1}{2}\sqrt{-3}}. \qquad (1)$$

$$B=\left[\frac{x^2+1}{(x-1)^4(x^2-x+1)}\right]_{x=-1}=\frac{1}{24},$$

$$A_1=\left[\frac{x^2+1}{(x+1)(x^2-x+1)}\right]_{x=1}=1,$$

$$C=\left[\frac{x^2+1}{(x-1)^4(x+1)(x-\frac{1}{2}+\frac{1}{2}\sqrt{-3})}\right]_{x=\frac{1}{2}+\frac{1}{2}\sqrt{-3}}=-\frac{1}{3},$$

$$D=\left[\frac{x^2+1}{(x-1)^4(x+1)(x-\frac{1}{2}-\frac{1}{2}\sqrt{-3})}\right]_{x=\frac{1}{2}-\frac{1}{2}\sqrt{-3}}=-\frac{1}{3}.$$

$$\frac{-\frac{1}{3}}{x-\frac{1}{2}-\frac{1}{2}\sqrt{-3}}+\frac{-\frac{1}{3}}{x-\frac{1}{2}+\frac{1}{2}\sqrt{-3}}=-\frac{1}{3}\cdot\frac{(2x-1)}{x^2-x+1}.$$

Substitute these values and clear (1) of fractions.

$$24(x^2+1)=24(x+1)(x^2-x+1)+24A_2(x-1)(x+1)(x^2-x+1)$$
$$+24A_3(x-1)^2(x+1)(x^2-x+1)+24A_4(x-1)^3(x+1)$$
$$(x^2-x+1)+(x-1)^4(x^2-x+1)-8(2x-1)(x-1)^4(x+1);$$

$$15x^6-51x^5+45x^4+6x^3-51x^2+45x-9=24A_2(x-1)(x+1)$$
$$(x^2-x+1)+24A_3(x-1)^2(x+1)(x^2-x+1)+24A_4$$
$$(x-1)^3(x+1)(x^2+x-1).$$

The second member of this equation is divisible by

$$(x-1)(x+1)(x^2-x+1),$$

therefore the first member must be divisible by the same quantity.

Dividing, we have

$$15x^2-36x+9=24A_2+24A_3(x-1)+24A_4(x-1)^2.$$

Let $x=1$, $\qquad -12=24A_2,$

$$A_2=-\frac{1}{2},$$

and we get

$$15x^2-36x+21=24A_3(x-1)+24A_4(x-1)^2.$$

Divide by $x-1$;

$$15x-21=24A_3+24A_4(x-1).$$

Let $x=1$, $\qquad -6=24A_3,$

$$A_3=-\frac{1}{4},$$

$$15x-15=24A_4(x-1).$$

Divide by $x-1$; $\qquad 15=24A_4,$

$$A_4=\frac{5}{8}.$$

Hence

$$\frac{x^2+1}{(x-1)^4(x^3+1)}=\frac{1}{(x-1)^4}-\frac{1}{2}\cdot\frac{1}{(x-1)^3}-\frac{1}{4}\cdot\frac{1}{(x-1)^2}+\frac{5}{8}\cdot\frac{1}{x-1}$$

$$+\frac{1}{24}\cdot\frac{1}{x+1}-\frac{1}{3}\cdot\frac{1}{x-\frac{1}{2}-\frac{1}{2}\sqrt{-3}}-\frac{1}{3}\cdot\frac{1}{x-\frac{1}{2}+\frac{1}{2}\sqrt{-3}}. \qquad (2)$$

62. Having shown that any rational fraction can be reduced to a sum of fractions which always come under one of the two forms $\dfrac{A}{(x-a)^n}$ and $\dfrac{A}{x-a}$, it remains to show that these forms can be integrated.

To find $\displaystyle\int\frac{Adx}{(x-a)^n}$,

let $$z=x-a,$$

then $$dz=dx,$$

and

$$\int\frac{Adx}{(x-a)^n}=A\int\frac{dz}{z^n}=-\frac{1}{(n-1)}\cdot\frac{A}{z^{n-1}}=-\frac{1}{(n-1)}\cdot\frac{A}{(x-a)^{n-1}}. \qquad [1]$$

To find $\displaystyle\int\frac{Adx}{x-a}$,

let $$z=x-a,$$

then $$dz=dx,$$

and $$\int\frac{Adx}{x-a}=A\int\frac{dz}{z}=A\log z=A\log(x-a). \qquad [2]$$

Turning back to Art. 58 (2), we find

$$\int\frac{(3x-1)\,dx}{(x-1)^2(x+1)}=\int\frac{dx}{(x-1)^2}+\int\frac{dx}{x-1}-\int\frac{dx}{x+1}=-\frac{1}{x-1}$$

$$+\log(x-1)-\log(x+1)=-\frac{1}{x-1}+\log\frac{x-1}{x+1}.$$

Turning to Art. 60 (1), we have

$$\int\frac{(3x-1)\,dx}{x(x-2)(x-1)}=\tfrac{1}{2}\int\frac{dx}{x}+\tfrac{5}{6}\int\frac{dx}{x-2}-\tfrac{4}{3}\int\frac{dx}{x+1}$$

$$=\tfrac{1}{2}\log x+\tfrac{5}{6}\log(x-2)-\tfrac{4}{3}\log(x+1).$$

63. If imaginary values come in when we break up our given fraction, they will disappear if we combine our results properly after integrating.

We know (Art. 28, Ex. 2) that if the denominator of our given fraction contains an imaginary factor, $(x-a-b\sqrt{-1})^n$, it will also contain the conjugate of that factor, namely, $(x-a+b\sqrt{-1})^n$. Moreover, since by Art. 59 the numerator of the partial fraction corresponding to $(x-a-b\sqrt{-1})^n$ will be the same rational algebraic function of $a+b\sqrt{-1}$ that the numerator of the partial fraction corresponding to $(x-a+b\sqrt{-1})^n$ is of $a-b\sqrt{-1}$, these two numerators must be conjugate imaginaries by Art. 28, Ex. 3. Hence, for every fraction of the form $\dfrac{A+B\sqrt{-1}}{(x-a-b\sqrt{-1})^n}$ we shall have a second of the form $\dfrac{A-B\sqrt{-1}}{(x-a+b\sqrt{-1})^n}$.

$$\int\frac{A+B\sqrt{-1}}{(x-a-b\sqrt{-1})^n}dx=-\frac{1}{(n-1)}\cdot\frac{(A+B\sqrt{-1})}{(x-a-b\sqrt{-1})^{n-1}},$$

by Art. 62 [1].

$$\int\frac{A-B\sqrt{-1}}{(x-a+b\sqrt{-1})^n}dx=-\frac{1}{(n-1)}\cdot\frac{(A-B\sqrt{-1})}{(x-a+b\sqrt{-1})^{n-1}}.$$

Let $$(x-a+b\sqrt{-1})^{n-1}=X+Y\sqrt{-1},$$

X and Y being real functions of x;

then $$(x-a-b\sqrt{-1})^{n-1}=X-Y\sqrt{-1}.$$

$$\int\frac{A+B\sqrt{-1}}{(x-a-b\sqrt{-1})^n}dx+\int\frac{A-B\sqrt{-1}}{(x-a+b\sqrt{-1})^n}dx$$

$$=-\frac{1}{(n-1)}\cdot\frac{(A+B\sqrt{-1})}{X-Y\sqrt{-1}}-\frac{1}{(n-1)}\cdot\frac{(A-B\sqrt{-1})}{X+Y\sqrt{-1}}$$

$$=-\frac{1}{(n-1)}\cdot\frac{(2AX+2BY)}{(x^2-2ax+a^2+b^2)^{n-1}}, \qquad [1]$$

a result which is free from imaginaries.

If $$n=1,$$

we have the pair of fractions, $\dfrac{A+B\sqrt{-1}}{x-a-b\sqrt{-1}}$ and $\dfrac{A-B\sqrt{-1}}{x-a+b\sqrt{-1}}$.

$$\int\frac{A+B\sqrt{-1}}{x-a-b\sqrt{-1}}dx=(A+B\sqrt{-1})\log(x-a-b\sqrt{-1}),$$

by Art. 62 [2],

$$\int\frac{A-B\sqrt{-1}}{x-a+b\sqrt{-1}}dx=(A-B\sqrt{-1})\log(x-a+b\sqrt{-1});$$

$$\log(x-a-b\sqrt{-1})=\tfrac{1}{2}\log[(x-a)^2+b^2]-\sqrt{-1}.\tan^{-1}\frac{b}{x-a},$$

$$\log(x-a+b\sqrt{-1})=\tfrac{1}{2}\log[(x-a)^2+b^2]+\sqrt{-1}.\tan^{-1}\frac{b}{x-a}.$$

Hence $$\int\frac{A+B\sqrt{-1}}{x-a-b\sqrt{-1}}dx+\int\frac{A-B\sqrt{-1}}{x-a+b\sqrt{-1}}dx$$

$$=A\log[(x-a)^2+b^2]+2B\tan^{-1}\frac{b}{x-a}, \qquad [2]$$

which is real.

The form of [2] can be modified by adding a constant.

$$\frac{\pi}{2}+\tan^{-1}\frac{b}{x-a}=\frac{\pi}{2}+\text{ctn}^{-1}\frac{x-a}{b}=\frac{\pi}{2}-\text{ctn}^{-1}\frac{a-x}{b}=\tan^{-1}\frac{a-x}{b}.$$

Hence $$A\log[(x-a)^2+b^2]+2B\tan^{-1}\frac{a-x}{b} \qquad [3]$$

differs from [2] by the constant $B\pi$, and therefore is a true value of $$\int\frac{A+B\sqrt{-1}}{x-a-b\sqrt{-1}}dx+\int\frac{A-B\sqrt{-1}}{x-a+b\sqrt{-1}}dx.$$

Turning back to Art. 61 (2) we find

$$\int\frac{x^2+1}{(x-1)^4(x^3+1)}dx=\int\frac{dx}{(x-1)^4}-\tfrac{1}{2}\int\frac{dx}{(x-1)^3}-\tfrac{1}{4}\int\frac{dx}{(x-1)^2}$$

$$+\tfrac{5}{8}\int\frac{dx}{x-1}+\tfrac{1}{24}\int\frac{dx}{x+1}-\tfrac{1}{3}\int\frac{dx}{x-\frac{1}{2}-\frac{1}{2}\sqrt{-3}}-\tfrac{1}{3}\int\frac{dx}{x-\frac{1}{2}+\frac{1}{2}\sqrt{-3}}$$

$$=-\frac{1}{3}\cdot\frac{1}{(x-1)^3}+\frac{1}{4}\cdot\frac{1}{(x-1)^2}+\frac{1}{4}\cdot\frac{1}{x-1}+\tfrac{5}{8}\log(x-1)$$

$$+\tfrac{1}{24}\log(x+1)-\tfrac{1}{3}\log(x^2-x+1).$$

EXAMPLES.

(1) $\displaystyle\int\frac{x^2-3x+3}{(x-1)(x-2)}\,dx = x+\log\frac{x-2}{x-1}.$

(2) $\displaystyle\int\frac{x^2-1}{x^2-4}\,dx = x+\tfrac{3}{4}\log\frac{x-2}{x+2}.$

(3) $\displaystyle\int\frac{dx}{x^2+1} = \tan^{-1}x.$

(4) $\displaystyle\int\frac{dx}{x^3-1} = \tfrac{1}{6}\log\frac{(x-1)^2}{x^2+x+1} - \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}.$

(5) $\displaystyle\int\frac{dx}{a^4-x^4} = \frac{1}{2a^3}\tan^{-1}\frac{x}{a} + \frac{1}{4a^3}\log\frac{a+x}{a-x}.$

(6) $\displaystyle\int\frac{dx}{(x^2+1)(x^2+x+1)} = \tfrac{1}{2}\log\frac{x^2+x+1}{x^2+1} + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}.$

(7) $\displaystyle\int\frac{x^2\,dx}{x^4+x^2-2} = \tfrac{1}{6}\log\frac{x-1}{x+1} + \frac{\sqrt{2}}{3}\tan^{-1}\frac{x}{\sqrt{2}}.$

(8) $\displaystyle\int\frac{x^2-1}{x^4+x^2+1}\,dx = \tfrac{1}{2}\log\frac{x^2-x+1}{x^2+x+1}.$

(9) $\displaystyle\int\frac{dx}{(x-1)^2(x^2+1)^2} = -\frac{1}{4(x-1)} - \tfrac{1}{2}\log(x-1)$

$\displaystyle\qquad + \tfrac{1}{4}\tan^{-1}x - \frac{1}{4(x^2+1)} + \tfrac{1}{4}\log(x^2+1).$

(10) $\displaystyle\int\frac{x^2\,dx}{x^4+1} = \frac{1}{4\sqrt{2}}\log\frac{x^2-x\sqrt{2}+1}{x_2+x\sqrt{2}+1}$

$\displaystyle\qquad + \frac{1}{2\sqrt{2}}\left[\tan^{-1}(x\sqrt{2}+1) + \tan^{-1}(x\sqrt{2}-1)\right].$

CHAPTER V.

REDUCTION FORMULAS.

64. The method given in the last chapter for the integration of rational fractions is open to the practical objection that it is often exceedingly laborious. In many cases much of the labor can be saved by making the required integration depend upon the integration of a simpler form. This is usually done by the aid of what is called a *reduction formula.*

Let the function to be integrated be of the form $x^{m-1}(a+bx^n)^p$, where m, n, and p may be positive or negative. If they are integers, the function in question is either an *algebraic polynomial* or a *rational fraction;* if they are fractions, the expression is irrational. The formulas we shall obtain will apply to either case.

Denote $a+bx^n$ by z; then we want $\int x^{m-1}z^p\,dx$.

Let $$z^p = u$$

and $$x^{m-1}dx = dv,\qquad \text{and } \textit{integrate by parts.}$$

$$du = pz^{p-1}dz = bnpx^{n-1}z^{p-1}dx,$$

$$v = \frac{x^m}{m},$$

$$\int x^{m-1}z^p\,dx = \frac{x^m z^p}{m} - \frac{bnp}{m}\int x^{m+n-1}z^{p-1}dx. \qquad [1]$$

This formula makes our integral depend upon the integral of an expression like the given one, except that the exponent of x has been increased while that of z has been decreased.

We get from [1], by transposition,

$$\int x^{m+n-1}z^{p-1}dx = \frac{x^m z^p}{bnp} - \frac{m}{bnp}\int x^{m-1}z^p\,dx.$$

Change $m+n$ into m and $p-1$ into p, whence m is changed into $m-n$ and p into $p+1$, and we get

$$\int x^{m-1}z^p\,dx = \frac{x^{m-n}z^{p+1}}{bn(p+1)} - \frac{m-n}{bn(p+1)}\int x^{m-n-1}z^{p+1}dx, \quad [2]$$

a formula that lowers the exponent of x while it raises that of z.

Since $$z = a + bx^n,$$

$$z^p = z^{p-1}(a+bx^n),$$

hence

$$\int x^{m-1}z^p\,dx = \int x^{m-1}z^{p-1}(a+bx^n)\,dx = a\int x^{m-1}z^{p-1}\,dx$$
$$+ b\int x^{m+n-1}z^{p-1}\,dx;$$

therefore, by [1],

$$\frac{x^m z^p}{m} - \frac{bnp}{m}\int x^{m+n-1}z^{p-1}dx = a\int x^{m-1}z^{p-1}dx + b\int x^{m+n-1}z^{p-1}dx,$$

$$\int x^{m-1}z^{p-1}\,dx = \frac{x^m z^p}{am} - \frac{b(m+np)}{am}\int x^{m+n-1}z^{p-1}dx.$$

Change p into $p+1$.

$$\int x^{m-1}z^p\,dx = \frac{x^m z^{p+1}}{am} - \frac{b(m+np+n)}{am}\int x^{m+n-1}z^p dx. \quad [3]$$

Change m into $m-n$, and transpose.

$$\int x^{m-1}z^p\,dx = \frac{x^{m-n}z^{p+1}}{b(m+np)} - \frac{a(m-n)}{b(m+np)}\int x^{m-n-1}z^p\,dx. \quad [4]$$

We have seen that

$$\int x^{m-1}z^p\,dx = a\int x^{m-1}z^{p-1}dx + b\int x^{m+n-1}z^{p-1}dx,$$

and, from [1],

$$b\int x^{m+n-1}z^{p-1}dx = \frac{x^m z^p}{np} - \frac{m}{np}\int x^{m-1}z^p\,dx;$$

hence

$$\int x^{m-1}z^p\,dx = a\int x^{m-1}z^{p-1}\,dx + \frac{x^m z^p}{np} - \frac{m}{np}\int x^{m-1}z^p\,dx,$$

$$\int x^{m-1}z^p\,dx = \frac{x^m z^p}{m+np} + \frac{anp}{m+np}\int x^{m-1}z^{p-1}\,dx. \qquad [5]$$

Change p into $p+1$, and transpose.

$$\int x^{m-1}z^p\,dx = -\frac{x^m z^{p+1}}{an(p+1)} + \frac{m+np+n}{an(p+1)}\int x^{m-1}z^{p+1}dx. \qquad [6]$$

Formula [3] enables us to raise, and formula [4] to lower, the exponent of x by n without affecting the exponent of z; while formula [5] enables us to lower, and formula [6] to raise, the exponent of z by unity without affecting the exponent of x.

Formulas [1] and [3] cannot be used when $m=0$;

formulas [2] and [6] cannot be used when $p=-1$;

formulas [4] and [5] cannot be used when $m=-np$;

for in all these cases infinite values will be brought into the second member of the formula.

65. If $n=1$, $\qquad z=a+bx,$

and our last four reduction formulas become

$$\int x^{m-1}z^p dx = \frac{x^m z^{p+1}}{am} - \frac{b(m+p+1)}{am}\int x^m z^p dx. \qquad [3]$$

$$\int x^{m-1}z^p dx = \frac{x^{m-1}z^{p+1}}{b(m+p)} - \frac{a(m-1)}{b(m+p)}\int x^{m-2}z^p dx. \qquad [4]$$

$$\int x^{m-1}z^p dx = \frac{x^m z^p}{m+p} + \frac{ap}{m+p}\int x^{m-1}z^{p-1}\,dx. \qquad [5]$$

$$\int x^{m-1}z^p dx = -\frac{x^m z^{p+1}}{a(p+1)} + \frac{m+p+1}{a(p+1)}\int x^{m-1}z^{p+1}dx. \qquad [6]$$

If m and p are integers, and $m>0$ and $p>0$, a repeated use of [5] will reduce p to zero, and we shall have to find merely the $\int x^{m-1}dx$.

If $m<0$ and $p>0$, [3] will enable us to raise m to 0, and then [5] will enable us to lower p to 0, and we shall need only $\int\frac{dx}{x}$.

If $m>0$ and $p<0$, [6] will raise p to -1, and [4] will then lower m to 1, and we shall need $\int\frac{dx}{z}$.

If $m<0$ and $p<0$, [6] will raise p to -1, and [3] will raise m to 0, and we shall need $\int\frac{dx}{xz}$.

$$\int x^{m-1}\,dx=\frac{x^m}{m},$$

$$\int\frac{dx}{x}=\log x,$$

$$\int\frac{dx}{z}=\int\frac{dx}{a+bx}=\frac{1}{b}\log(a+bx),$$

$$\int\frac{dx}{xz}=\int\frac{dx}{x(a+bx)}=-\frac{1}{a}\log\frac{a+bx}{x}.$$

Hence, when $n=1$, and m and p are integers, our reduction formulas always lead to the desired result.

EXAMPLES.

(1) $\displaystyle\int\frac{dx}{x^5(a+bx)}=-\frac{b^4}{a^5}\log\frac{a+bx}{x}+\frac{b^3}{a^4x}-\frac{b^2}{2a^3x^2}+\frac{b}{3a^2x^3}-\frac{1}{4ax^4}.$

(2) Consider the case where $n=2$, rewriting the reduction formulas to suit the case, and giving an exhaustive investigation.

(3) $$\int\frac{x^2\,dx}{(a+bx^2)^3}=-\frac{x}{4b(a+bx^2)^2}+\frac{x}{8ab(a+bx^2)}+\frac{1}{8(ab)^{\frac{3}{2}}}\tan^{-1}x\sqrt{\frac{b}{a}}.$$

CHAPTER VI.

IRRATIONAL FORMS.

66. We have seen that algebraic polynomials and rational fractions can always be integrated. When we come to irrational expressions, however, very few forms are integrable, and most of these have to be rationalized by ingenious substitutions.

If an algebraic function is irrational because of the presence of an expression of the first degree under the radical sign, it can be easily made rational.

Let $f(x, \sqrt[n]{a+bx})$ be the function in question.

Let $$z = \sqrt[n]{a+bx};$$

then $$z^n = a + bx.$$

$$nz^{n-1}dz = bdx,$$

$$dx = \frac{nz^{n-1}dz}{b};$$

$$x = \frac{z^n - a}{b}.$$

Hence $$\int f(x, \sqrt[n]{a+bx})dx = \frac{n}{b}\int f\left(\frac{z^n-a}{b}, z\right)z^{n-1}dz,$$

which is rational and can be treated by the methods of Chapter IV.

EXAMPLES.

(1) $\displaystyle\int \frac{\sqrt{x}+1}{\sqrt{x}-1}dx = x + 4\sqrt{x} + 4\log(\sqrt{x}-1).$

(2) $\displaystyle\int \sqrt[n]{(ax+b)^m}\,dx = \frac{n\sqrt[n]{(ax+b)^{m+n}}}{a(m+n)}.$

(3) $\displaystyle\int [x\sqrt[n]{(x+a)} + \sqrt{(x+a)}]dx$

$$= \frac{n\sqrt[n]{(x+a)^{2n+1}}}{2n+1} - \frac{na\sqrt[n]{(x+a)^{n+1}}}{n+1} + \tfrac{2}{3}\sqrt{(x+a)^3}.$$

67. A case not unlike the last is $\int f(x, \sqrt[n]{c+\sqrt[m]{a+bx}})\,dx$.

Let
$$z = \sqrt[n]{c+\sqrt[m]{a+bx}};$$
$$z^n = c+\sqrt[m]{a+bx},$$
$$(z^n - c)^m = a+bx,$$
$$x = \frac{(z^n-c)^m - a}{b},$$
$$dx = \frac{mn(z^n-c)^{m-1}z^{n-1}dz}{b}.$$

Hence
$$\int f(x, \sqrt[n]{c+\sqrt[m]{a+bx}})\,dx$$
$$= \frac{mn}{b}\int f\left[\frac{(z^n-c)^m - a}{b}, z\right](z^n-c)^{m-1}z^{n-1}dz.$$

EXAMPLES.

(1) Find $\int \frac{xdx}{\sqrt{c+\sqrt{a+bx}}}$.

(2) Find $\int \frac{dx}{\sqrt[4]{1+\sqrt{1-x}}}$.

68. If the expression under the radical is of a higher degree than the first the function cannot in general be rationalized. The only important exceptional case is where the function to be integrated is irrational by reason of containing the square root of a quantity of the second degree.

Required $\int f(x, \sqrt{a+bx+cx^2})\,dx$.

First Method. Let c be positive; take out $\sqrt{c}$ as a factor, and the radical may be written $\sqrt{A+Bx+x^2}$.

Let
$$\sqrt{A+Bx+x^2} = x+z,$$
$$A+Bx+x^2 = x^2+2xz+z^2,$$
$$x = \frac{z^2-A}{B-2z},$$

$$dx = -\frac{2(z^2 - Bz + A)dz}{(B-2z)^2},$$

$$\sqrt{A + Bx + x^2} = x + z = -\frac{z^2 - Bz + A}{B - 2z},$$

and the substitution of these values will render the given function rational.

Second Method. Let c be positive; take out $\sqrt{c}$ as a factor, and, as before, the radical may be written $\sqrt{A + Bx + x^2}$.

Let $$\sqrt{A+Bx+x^2} = \sqrt{A} + xz;$$

$$A + Bx + x^2 = A + 2\sqrt{A}\,.\,xz + x^2z^2,$$

$$x = \frac{2\sqrt{A}\,.\,z - B}{1 - z^2},$$

$$dx = \frac{2(\sqrt{A}\,.\,z^2 - Bz + \sqrt{A})dz}{(1-z^2)^2},$$

$$\sqrt{A+Bx+x^2} = \sqrt{A} + xz = \frac{\sqrt{A}\,.\,z^2 - Bz + \sqrt{A}}{1 - z^2},$$

and the substitution of these values will render the given function rational.

If c is negative the radical can be reduced to the form $\sqrt{A + Bx - x^2}$, and the method just given will present no difficulty.

Third Method. Let c be positive; the radical will reduce to $\sqrt{A + Bx + x^2}$. Resolve the quantity under the radical into the product of two binomial factors $(x - \alpha)(x - \beta)$, α and β being the roots of the equation $A + Bx + x^2 = 0$.

Let $$\sqrt{(x-\alpha)(x-\beta)} = (x - \alpha)z;$$

$$(x-\alpha)(x-\beta) = (x-\alpha)^2z^2,$$

$$x = \frac{\beta - \alpha z^2}{1 - z^2},$$

$$dx = \frac{2z(\beta - \alpha)dz}{(1-z^2)^2},$$

$$\sqrt{(x-\alpha)(x-\beta)} = (x-\alpha)z = \frac{(\beta-\alpha)z}{1 - z^2},$$

and the substitution of these values will make the given function rational.

If c is negative the radical will reduce to $\sqrt{A + Bx - x^2}$, and may be written $\sqrt{(\alpha - x)(x - \beta)}$ where α and β are the roots of $x^2 - Bx - A = 0$, and the method just explained will apply.

In general, that one of the three methods is preferable which will avoid introducing imaginary constants; the first, if $c > 0$; the second, if $c < 0$ and $\frac{a}{-c} > 0$; the third, if $c < 0$ and $\frac{a}{-c} < 0$. If the roots α and β are imaginary, and $A = \frac{a}{-c}$ is negative, it will be impossible to avoid imaginaries, for in that case $A + Bx - x^2$ will be negative for all real values of x.

69. Let us compare the working of the three methods just given by applying them in turn to the example $\int \frac{dx}{\sqrt{2 + 3x + x^2}}$.

1st. Let $\sqrt{2 + 3x + x^2} = x + z$;

$$\int \frac{dx}{\sqrt{2 + 3x + x^2}} = \int \frac{2(z^2 - 3z + 2)\,dz}{(3 - 2z)^2} \cdot \frac{3 - 2z}{z^2 - 3z + 2} = \int \frac{2\,dz}{3 - 2z}$$

$$= -\log(3 - 2z),$$

$$\int \frac{dx}{\sqrt{2 + 3x + x^2}} = -\log(3 + 2x - 2\sqrt{2 + 3x + x^2})$$

$$= \log \frac{1}{3 + 2x - 2\sqrt{2 + 3x + x^2}}$$

$$= \log \frac{3 + 2x + 2\sqrt{2 + 3x + x^2}}{9 + 12x + 4x^2 - 8 - 12x - 4x^2}$$

$$= \log[3 + 2x + 2\sqrt{2 + 3x + x^2}]. \qquad (1)$$

2d. Let $\sqrt{2 + 3x + x^2} = \sqrt{2} + xz$;

$$\int \frac{dx}{\sqrt{2 + 3x + x^2}} = 2\int \frac{(\sqrt{2}.z^2 - 3z + \sqrt{2})\,dz}{(1 - z^2)^2} \cdot \frac{1 - z^2}{\sqrt{2}.z^2 - 3z + \sqrt{2}}$$

$$= 2\int \frac{dz}{1 - z^2} = \log \frac{1 + z}{1 - z}. \qquad \text{(Art. 52)}$$

$$\int\frac{dx}{\sqrt{2+3x+x^2}}=\log\frac{x-\sqrt{2}+\sqrt{2+3x+x^2}}{x+\sqrt{2}-\sqrt{2+3x+x^2}}$$

$$=\log\frac{x^2+2x\sqrt{2+3x+x^2}+2+3x+x^2-2}{x^2+2\sqrt{2}.x+2-2-3x-x^2}$$

$$=\log\frac{3+2x+2\sqrt{2+3x+x^2}}{2\sqrt{2}-3}$$

$$=\log(3+2x+2\sqrt{2+3x+x^2})-\log(2\sqrt{2}-3),$$

or, dropping the constant $\log(2\sqrt{2}-3)$,

$$\int\frac{dx}{\sqrt{2+3x+x^2}}=\log(3+2x+2\sqrt{2+3x+x^2}). \tag{2}$$

3d. Let $\sqrt{2+3x+x^2}=\sqrt{(x+1)(x+2)}=(x+1)z$;

$$\int\frac{dx}{\sqrt{2+3x+x^2}}=2\int\frac{-zdz}{(1-z^2)^2}\,\frac{1-z^2}{-z}=2\int\frac{dz}{1-z^2}=\log\frac{1+z}{1-z}.$$

$$\int\frac{dx}{\sqrt{2+3x+x^2}}=\log\frac{1+\sqrt{\dfrac{x+2}{x+1}}}{1-\sqrt{\dfrac{x+2}{x+1}}}=\log\frac{\sqrt{x+1}+\sqrt{x+2}}{\sqrt{x+1}-\sqrt{x+2}}$$

$$=\log\frac{x+1+2\sqrt{2+3x+x^2}+x+2}{x+1-x-2}$$

$$=\log(3+2x+2\sqrt{2+3x+x^2})+\log(-1),$$

or, dropping the imaginary constant $\log(-1)$,

$$\int\frac{dx}{\sqrt{2+3x+x^2}}=\log(3+2x+2\sqrt{2+3x+x^2}). \tag{3}$$

Examples.

(1) $\displaystyle\int\frac{dx}{(2+3x)\sqrt{4-x^2}}=\frac{1}{4\sqrt{2}}\log\frac{\sqrt{4+2x}-\sqrt{2-x}}{\sqrt{4+2x}+\sqrt{2-x}}.$

(2) $\displaystyle\int\frac{dx}{\sqrt{x^2+x}}=\log(\tfrac{1}{2}+x+\sqrt{x^2+x}).$

(3) $\displaystyle\int\frac{dx}{\sqrt{a+bx+cx^2}}=\frac{1}{\sqrt{c}}\log\left(\frac{b}{2\sqrt{c}}+x\sqrt{c}+\sqrt{a+bx+cx^2}\right).$

70. If the function is irrational through the presence, under the radical sign, of a fraction whose numerator and denominator are of the first degree, it can always be rationalized.

Required $\int f\left(x, \sqrt[n]{\dfrac{ax+b}{lx+m}}\right) dx.$

Let
$$z = \sqrt[n]{\frac{ax+b}{lx+m}},$$
$$z^n = \frac{ax+b}{lx+m},$$
$$x = \frac{b - mz^n}{lz^n - a},$$
$$dx = \frac{n(am - bl)z^{n-1}dz}{(lz^n - a)^2},$$

and the substitution of these values will make the given function rational.

EXAMPLE.

$$\int \frac{dx}{(1+x)^2}\sqrt[3]{\frac{1-x}{1+x}} = -\tfrac{3}{8}\sqrt[3]{\left(\frac{1-x}{1+x}\right)^4}.$$

71. If the function to be integrated is of the form $x^{m-1}(a+bx^n)^p$, m, n, and p being any numbers positive or negative, and one at least of them being fractional, the reduction formulas of Art. 64 will often lead to the desired integral.

EXAMPLES.

(1) $\displaystyle\int \frac{x^4\,dx}{(1-x^2)^{\frac{1}{2}}} = \tfrac{3}{8}\sin^{-1}x - \frac{x\sqrt{1-x^2}}{8}(3+2x^2).$

(2) $\displaystyle\int \frac{dx}{x^3\sqrt{1-x^2}} = \tfrac{1}{2}\log\frac{1-\sqrt{1-x^2}}{x} - \frac{\sqrt{1-x^2}}{2x^2}.$

(3) $\displaystyle\int \frac{x^2\,dx}{(2ax-x^2)^{\frac{1}{2}}} = -(2ax-x^2)^{\frac{1}{2}}\left(\frac{x}{2}+\frac{3a}{2}\right) + 3a^2\sin^{-1}\sqrt{\frac{x}{2a}}.$

(4) $\displaystyle\int \frac{x^3\,dx}{(a^2+x^2)^{\frac{5}{2}}} = -\frac{(2a^2+3x^2)}{3(a^2+x^2)^{\frac{3}{2}}}.$

72. We have said that when an irrational function contains a quantity of a higher degree than the second, under the radical sign, it cannot *ordinarily* be integrated. It would be more correct to say that its integral cannot ordinarily be finitely expressed in terms of the functions with which we are familiar.

The integrals of a large class of such irrational expressions have been specially studied under the name of Elliptic Functions. They have peculiar properties, and can be expressed in terms of ordinary functions only by the aid of infinite series.

CHAPTER VII.

TRANSCENDENTAL FUNCTIONS.

73. In dealing with the integration of transcendental functions the method of *integration by parts* is generally the most effective.

For example. Required $\int x(\log x)^2 dx$.

Let
$$u = (\log x)^2,$$
$$dv = x.dx;$$
$$du = \frac{2\log x.dx}{x},$$
$$v = \frac{x^2}{2},$$

$$\int x(\log x)^2 = \frac{x^2(\log x)^2}{2} - \int x\log x.dx = \frac{x^2}{2}\left[(\log x)^2 - \log x + \tfrac{1}{2}\right].$$

Again. Required $\int e^x \sin x.dx$.

$$u = \sin x,$$
$$dv = e^x dx;$$
$$du = \cos x.dx,$$
$$v = e^x,$$
$$\int e^x \sin x.dx = e^x \sin x - \int e^x \cos x.dx,$$
$$\int e^x \cos x.dx = e^x \cos x + \int e^x \sin x.dx;$$

whence
$$\int e^x \sin x.dx = \frac{e^x(\sin x - \cos x)}{2},$$

and
$$\int e^x \cos x.dx = \frac{e^x(\sin x + \cos x)}{2}.$$

EXAMPLES.

(1) $$\int x^m (\log x)^3 dx = \frac{x^{m+1}}{m+1}\left[(\log x)^3 - \frac{3(\log x)^2}{m+1} + \frac{6\log x}{(m+1)^2} - \frac{6}{(m+1)^3}\right].$$

(2) $$\int \frac{\log x.dx}{(1-x)^2} = \frac{x\log x}{1+x} + \log(1-x).$$

(3) $$\int e^{ax}\sqrt{(1-e^{2ax})}.dx = \frac{1}{2a}\left[e^{ax}\sqrt{(1-e^{2ax})} + \sin^{-1}e^{ax}\right].$$

74. The method of integration by parts gives us important reduction formulas for transcendental functions. Let us consider $\int \sin^n x.dx$.

$$u = \sin^{n-1}x,$$
$$dv = \sin x.dx;$$
$$du = (n-1)\sin^{n-2}x\cos x.dx,$$
$$v = -\cos x;$$

$$\int \sin^n x.dx = -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\cos^2 x.dx$$
$$= -\sin^{n-1}x\cos x + (n-1)\int(\sin^{n-2}x - \sin^n x)dx;$$

$$\int \sin^n x.dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x.dx. \qquad [1]$$

Transposing, and changing n into $n+2$, we get

$$\int \sin^n x.dx = \frac{1}{n+1}\sin^{n+1}x\cos x + \frac{n+2}{n+1}\int \sin^{n+2}x.dx. \qquad [2]$$

In like manner we get

$$\int \cos^n x.dx = \frac{1}{n}\sin x\cos^{n-1}x + \frac{n-1}{n}\int \cos^{n-2}x.dx, \qquad [3]$$

$$\int \cos^n x.dx = -\frac{1}{n+1}\sin x\cos^{n+1}x + \frac{n+2}{n+1}\int \cos^{n+2}x.dx. \qquad [4]$$

If n is a positive integer, formulas [1] and [3] will enable us to reduce the exponent of the sine or cosine to one or to zero,

and then we can integrate by inspection. If n is a negative integer, formulas [2] and [4] will enable us to raise the exponent to zero or to minus one. In the latter case we shall need $\int\frac{dx}{\cos x}$, or $\int\frac{dx}{\sin x}$, which have been found in Art. 54 (c).

EXAMPLES.

(1) $$\int \sin^4 x.dx = -\frac{\sin x \cos x}{4}\left(\sin^2 x + \frac{3}{2}\right) + \frac{3}{8}x.$$

(2) $$\int \cos^6 x.dx = \frac{\sin x \cos^5 x}{6}\left(\cos^2 x + \frac{5}{4}\right) + \frac{5}{16}(\sin x \cos x + x).$$

(3) $$\int \frac{dx}{\sin^3 x} = -\frac{\cos x}{2\sin^2 x} + \tfrac{1}{2}\log\tan\frac{x}{2}.$$

(4) Obtain the formulas

$$\int \mathrm{Sh}^n x.dx = \frac{1}{n}\mathrm{Sh}^{n-1}x\,\mathrm{Ch}\,x - \frac{n-1}{n}\int \mathrm{Sh}^{n-2}x.dx. \qquad [1]$$

$$\int \mathrm{Sh}^n x.dx = \frac{1}{n+1}\mathrm{Sh}^{n+1}x\,\mathrm{Ch}\,x - \frac{n+2}{n+1}\int \mathrm{Sh}^{n+2}x.dx. \qquad [2]$$

$$\int \mathrm{Ch}^n x.dx = \frac{1}{n}\,\mathrm{Sh}\,x\,\mathrm{Ch}^{n-1}x + \frac{n-1}{n}\int \mathrm{Ch}^{n-2}x.dx. \qquad [3]$$

$$\int \mathrm{Ch}^n x.dx = -\frac{1}{n+1}\,\mathrm{Sh}\,x\,\mathrm{Ch}^{n+1}x + \frac{n+2}{n+1}\int \mathrm{Ch}^{n+2}x.dx. \qquad [4]$$

(5) $$\int \frac{dx}{\mathrm{Sh}^3 x} = -\tfrac{1}{2}\frac{\mathrm{Ch}\,x}{\mathrm{Sh}^2 x} - \tfrac{1}{4}\log\frac{\mathrm{Ch}\,x - 1}{\mathrm{Ch}\,x + 1}.$$

75. The $(\sin^{-1}x)^n dx$ can be integrated by the aid of a reduction formula.

Let $$z = \sin^{-1}x;$$

then $$x = \sin z,$$

$$dx = \cos z.dz,$$

and $$\int (\sin^{-1}x)^n dx = \int z^n \cos z.dz.$$

Let
$$u = z^n,$$
$$dv = \cos z.dz;$$
$$du = nz^{n-1}dz,$$
$$v = \sin z;$$
$$\int z^n \cos z.dz = z^n \sin z - n\int z^{n-1}\sin z.dz.$$

$\int z^{n-1}\sin z.dz$ can be reduced in the same way, and is equal to $-z^{n-1}\cos z + (n-1)\int z^{n-2}\cos z.dz$;
hence
$$\int z^n \cos z.dz = z^n \sin z + nz^{n-1}\cos z - n(n-1)\int z^{n-2}\cos z.dz, \quad [1]$$
or
$$\int (\sin^{-1}x)^n dx = x(\sin^{-1}x)^n + n\sqrt{1-x^2}(\sin^{-1}x)^{n-1}$$
$$- n(n-1)\int (\sin^{-1}x)^{n-2}dx. \quad [2]$$

If n is a positive integer, this will enable us to make our required integral depend upon $\int dx$ or $\int \sin^{-1}x.dx$, the latter of which forms has been found in (I. Art. 81).

EXAMPLES.

(1) Obtain a formula for $\int (\text{vers}^{-1}x)^n dx$.

(2) $$\int (\sin^{-1}x)^4 dx = x[(\sin^{-1}x)^4 - 4.3.(\sin^{-1}x)^2 + 4.3.2.1]$$
$$+ 4\sqrt{1-x^2}\sin^{-1}x[(\sin^{-1}x)^2 - 3.2].$$

76. Integration by substitution is sometimes a valuable method in dealing with transcendental forms, and in the case of the trigonometric functions often enables us to reduce the given form to an algebraic one. Let it be required to find $\int (f\sin x)\cos x.dx$.

Let
$$z = \sin x,$$
$$dz = \cos x.dx;$$
$$\int (f\sin x)\cos x.dx = \int fz.dz.$$

In the same way we see that

$$\int (f\cos x)\sin x.dx = -\int fz.dz \qquad \text{if } z=\cos x,$$

and

$$\int [f(\sin x,\ \cos x)]\cos x.dx = \int [f(z,\ \sqrt{1-z^2})]dz \ \text{ if } z=\sin x,$$

$$\int [f(\sin x,\ \cos x)]\sin x.dx = -\int [f(z,\ \sqrt{1-z^2})]dz \ \text{ if } z=\cos x.$$

77. $\int \sin^m x \cos^n x.dx$ can be readily found by the method of Art. 76 if m and n are positive integers, and if either of them is odd. Let n be odd, then

$$\cos^n x = \cos^{n-1} x \cos x = (1-\sin^2 x)^{\frac{n-1}{2}} \cos x,$$

$$\int \sin^m x \cos^n x.dx = \int \sin^m x\,(1-\sin^2 x)^{\frac{n-1}{2}} \cos x.dx.$$

Let $$z = \sin x,$$

$$dz = \cos x.dx,$$

$$\int \sin^m x \cos^n x.dx = \int z^m\,(1-z^2)^{\frac{n-1}{2}} dz,$$

which can be expanded into an algebraic polynomial and integrated directly.

If m and n are positive integers, and are both even,

$$\int \sin^m x \cos^n x.dx = \int \sin^m x (1-\sin^2 x)^{\frac{n}{2}} dx.$$

$\sin^m x\,(1-\sin^2 x)^{\frac{n}{2}}$ can be expanded and thus integrated by Art. 74 [1].

If m or n is negative, and odd, we can write

$$\cos^n x = \cos^{n-1} x \cos x, \quad \text{or} \quad \sin^m x = \sin^{m-1} x \sin x,$$

and reduce the function to be integrated to a rational fraction by the substitution of

$$z = \cos x, \quad \text{or} \quad z = \sin x.$$

$\int \sin^m x \cos^n x.dx$ can also be treated by the aid of reduction formulas easily obtained.

EXAMPLES.

(1) $$\int \sin^3 x \cos^7 x . dx = \frac{\cos^{10} x}{10} - \frac{\cos^8 x}{8}.$$

(2) $$\int \cos^3 x \sqrt{\sin x} . dx = \frac{2 \sin^{\frac{3}{2}} x}{3} - \frac{2 \sin^{\frac{7}{2}} x}{7}.$$

(3) $$\int \frac{\sin^3 x . dx}{\sqrt{\cos x}} = \frac{2 \cos^{\frac{5}{2}} x}{5} - 2 \cos^{\frac{1}{2}} x.$$

(4) $$\int \cos^2 x \sin^4 x . dx = \frac{\sin x \cos x}{2} \left(\frac{\sin^4 x}{3} - \frac{\sin^2 x}{12} - \frac{1}{8} \right) + \frac{x}{16}.$$

(5) $$\int \frac{dx}{\sin x \cos^2 x} = \sec x + \log \tan \frac{x}{2}.$$

(6) $$\int \frac{dx}{\sin^3 x \cos^2 x} = \sec x - \frac{\cos x}{2 \sin^2 x} + \frac{3}{2} \log \tan \frac{x}{2}.$$

(7) $$\int \frac{dx}{\tan^5 x} = -\frac{1}{4 \tan^4 x} + \frac{1}{2 \tan^2 x} + \log \sin x.$$

CHAPTER VIII.

DEFINITE INTEGRALS.

78. A definite integral has been defined as the limit of a sum of infinitesimals, and we have proved that if the function to be integrated is continuous between the values between which the sum is to be taken, this limit can be found by taking the difference between two values of an indefinite integral.

In some cases it is possible to find the value of a definite integral from elementary considerations without using the indefinite integral, and it is worth while to take one or two examples where this can be done.

Required $\int_0^\pi \cos^3 x.dx.$

By our definition this must equal

$$\begin{aligned}
&\text{limit}\left[\cos^3 0.dx + \cos^3 dx.dx + \cos^3 2\,dx.dx + \cos^3 3\,dx.dx + \cdots\cdots\right.\\
&\qquad \left.+ \cos^3(\pi - 3\,dx).dx + \cos^3(\pi - 2\,dx).dx + \cos^3(\pi - dx).dx\right]\\
&\qquad = \text{limit}\left[dx + \cos^3 dx.dx + \cos^3 2\,dx.dx + \cos^3 3\,dx.dx + \cdots\cdots\right.\\
&\qquad \left.- \cos^3 3\,dx.dx - \cos^3 2\,dx.dx - \cos^3 dx.dx\right],
\end{aligned}$$

since $\cos(\pi - \phi) = -\cos\phi$.

We see that in this sum the terms destroy each other in pairs, with the exception of the first term dx if the number of terms is odd, and with the exception of the first term and a term $\cos^3 \frac{n}{2} dx.dx$ in the middle of the set, if n is even. Each of these terms has zero for its limit as dx approaches zero; hence

$$\int_0^\pi \cos^3 x.dx = 0.$$

79. Required $\int_0^{\pi} \sin^2 x.dx$.

$$\int_0^{\pi} \sin^2 x.dx$$

$$= \lim [\sin^2 0.dx + \sin^2 dx.dx + \sin^2 2\,dx.dx + \sin^2 3\,dx.dx + \cdots\cdots$$
$$+ \sin^2(\pi - 3\,dx).dx + \sin^2(\pi - 2\,dx).dx + \sin^2(\pi - dx).dx]$$
$$= \lim [0 + 2\sin^2 dx.dx + 2\sin^2 2\,dx.dx + 2\sin^2 3\,dx.dx + \cdots\cdots]$$
$$= 2\int_0^{\frac{\pi}{2}} \sin^2 x.dx = 2\lim [\sin^2 0.dx + \sin^2 dx.dx + \sin^2 2\,dx.dx + \cdots\cdots$$
$$+ \sin^2\left(\frac{\pi}{2} - 2\,dx\right).dx + \sin^2\left(\frac{\pi}{2} - dx\right).dx]$$
$$= 2\lim [\sin^2 0.dx + \sin^2 dx.dx + \sin^2 2\,dx.dx + \sin^2 3\,dx.dx + \cdots\cdots$$
$$+ \cos^2 3\,dx.dx + \cos^2 2\,dx.dx + \cos^2 dx.dx]$$
$$= 2\lim [dx + dx + dx + \cdots\cdots].$$

Since $\qquad \sin^2\phi + \cos^2\phi = 1,$

but $\qquad dx + dx + dx + \cdots\cdots = \frac{\pi}{4}.$

Hence $\qquad \int_0^{\pi} \sin^2 x.dx = \frac{2\pi}{4} = \frac{\pi}{2}.$

EXAMPLES.

(1) Confirm these results by obtaining $\int_0^{\pi} \cos^3 x.dx$ and $\int_0^{\pi} \sin^2 x.dx$ by the usual methods.

(2) Find by elementary methods $\int_0^{2\pi} \sin^5 x.dx$.

80. It is generally necessary, however, to obtain a required definite integral by substituting in the value of the indefinite integral according to Art. 78, and this can always be done when the function to be integrated is finite and continuous between the values between which the definite integral is to be taken; that is, between what are called the *limits of integration*.

EXAMPLES.

(1) $\displaystyle\int_0^{\frac{\pi}{4}}\frac{\sin\theta.d\theta}{\cos^2\theta}$. *Ans.* $\sqrt{2}-1$.

(2) $\displaystyle\int_0^{a}\frac{dx}{\sqrt{x+a}+\sqrt{x}}$. *Ans.* $\frac{4}{3}\sqrt{a}(\sqrt{2}-1)$.

(3) $\displaystyle\int_0^{\infty}\frac{dx}{a^2+x^2}$. *Ans.* $\dfrac{\pi}{2a}$.

(4) $\displaystyle\int_0^{a}\frac{dx}{\sqrt{a^2-x^2}}$. *Ans.* $\dfrac{\pi}{2}$.

(5) $\displaystyle\int_0^{\infty}e^{-ax}dx$ (a positive). *Ans.* $\dfrac{1}{a}$.

(6) $\displaystyle\int_0^{\infty}e^{-ax}\sin mx.dx$. *Ans.* $\dfrac{m}{a^2+m^2}$.

(7) $\displaystyle\int_0^{\infty}e^{-ax}\cos mx.dx$. *Ans.* $\dfrac{a}{a^2+m^2}$.

81. When we have occasion to use a reduction formula in finding a definite integral, it is often worth while to substitute the limits of integration in the general formula before attempting to find the indefinite integral.

For example, let us find $\displaystyle\int_0^{a}\frac{x^6dx}{\sqrt{a^2-x^2}}$.

We can reduce the exponent of x by Art. 64 [4]

$$\int x^{m-1}z^p\,dx=\frac{x^{m-n}z^{p+1}}{b(m+np)}-\frac{a(m-n)}{b(m+np)}\int x^{m-n-1}z^p\,dx. \qquad [4]$$

For our example this becomes

$$\int x^{m-1}(a^2-x^2)^{-\frac{1}{2}}dx$$

$$=\frac{x^{m-2}(a^2-x^2)^{\frac{1}{2}}}{-m+1}-\frac{a^2(m-2)}{-m+1}\int x^{m-3}(a^2-x^2)^{-\frac{1}{2}}dx.$$

When $x=0$, $\dfrac{x^{m-2}(a^2-x^2)^{\frac{1}{2}}}{-m+1}=0$, and also when $x=a$.

Hence

$$\int_0^a x^{m-1}(a^2-x^2)^{-\frac{1}{2}}dx = \frac{a^2(m-2)}{m-1}\int_0^a x^{m-3}(a^2-x^2)^{-\frac{1}{2}}dx.$$

$$\int_0^a x^6(a^2-x^2)^{-\frac{1}{2}}dx = \frac{5a^2}{6}\int_0^a x^4(a^2-x^2)^{-\frac{1}{2}}dx$$

$$= \frac{5}{6}\cdot\frac{3}{4}a^4\int_0^a x^2(a^2-x^2)^{-\frac{1}{2}}dx$$

$$= +\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}a^6\int_0^a \frac{dx}{\sqrt{a^2-x^2}};$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a},$$

$$\int_0^a \frac{dx}{\sqrt{a^2-x^2}} = \left[\sin^{-1}\frac{x}{a}\right]_{x=a} - \left[\sin^{-1}\frac{x}{a}\right]_{x=0} = \sin^{-1}(1) - \sin^{-1}(0).$$

$\sin^{-1}(1) = \frac{\pi}{2}$ or $\frac{\pi}{2} + 2\pi$ or $\frac{\pi}{2} + 4\pi$. In general, $\frac{\pi}{2} + 2n\pi$.

$\sin^{-1}(0) = 0$ or 2π or 4π. In general, $2n\pi$.

If we take $2n\pi$ as the value $\sin^{-1}$ has when $x = 0$, and then increase x to 1, as all the increments of the $\sin^{-1}$ $\frac{dx}{\sqrt{a^2-x^2}}$ are positive, our whole increment must be positive, and we must take $2n\pi + \frac{\pi}{2}$ as the value of $\sin^{-1}(1)$. Hence

$$\sin^{-1}(1) - \sin^{-1}0 = \frac{\pi}{2} \quad\text{and}\quad \int_0^a \frac{x^6\,dx}{\sqrt{a^2-x^2}} = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\,\frac{\pi a^6}{2}.$$

EXAMPLES.

(1) $\displaystyle\int_0^a \frac{x^5\,dx}{\sqrt{a^2-x^2}} = \frac{2}{3}\cdot\frac{4}{5}a^5.$

(2) $\displaystyle\int_0^a \sqrt{a^2-x^2}.dx = \frac{\pi a^2}{4}.$

(3) $\displaystyle\int_0^a x^2\sqrt{a^2-x^2}.dx = \frac{1}{4}\cdot\frac{\pi a^4}{4}.$

(4) $\int_0^a x^2(a^2 - x^2)^{\frac{3}{2}} dx = \frac{1}{6}\cdot\frac{3}{16}\pi a^6.$

(5) Show that $\int_0^{\frac{\pi}{2}} \sin^n x.dx = \frac{1.3.5\ldots\ldots(n-1)}{2.4.6\ldots\ldots n}\frac{\pi}{2}$ when n is even

$= \frac{2.4.6\ldots\ldots(n-1)}{3.5.7\ldots\ldots n}$ when n is odd.

(6) Show that $\int_0^{\frac{\pi}{2}} \cos^n x.dx = \int_0^{\frac{\pi}{2}} \sin^n x.dx.$

(7) $\int_0^1 \frac{x^{2n}\,dx}{\sqrt{1-x^2}} = \frac{1.3.5\ldots\ldots(2n-1)}{2.4.6\ldots\ldots 2n}\frac{\pi}{2}.$

(*Suggestion:* Let $x = \sin\theta$.)

82. Required $\int_0^\infty e^{-x}x^n\,dx.$

$\int e^{-x}x^n\,dx = -e^{-x}x^n + n\int e^{-x}x^{n-1}dx$, by *integration by parts.*

When $x = 0$, $\frac{x^n}{e^x} = 0$; but when $x = \infty$, $\frac{x^n}{e^x} = \frac{\infty}{\infty}$, and is indeterminate. Determining it by the method of I. Art. 141, we find that its *true value* is 0; hence

$$\int_0^\infty e^{-x}x^n\,dx = n\int_0^\infty e^{-x}x^{n-1}\,dx = n(n-1)\int_0^\infty e^{-x}x^{n-2}\,dx. \qquad (1)$$

If n is an integer, this gives us

$$\int_0^\infty e^{-x}x^n\,dx = n\,!\int_0^\infty e^{-x}\,dx.$$

$\int e^{-x}dx = -e^{-x} = -\frac{1}{e^x}$, which is equal to 0 when $x = \infty$, and to -1 when $x = 0$,

hence $$\int_0^\infty e^{-x}\,dx = 1,$$

and we have $$\int_0^\infty e^{-x}x^n\,dx = n\,!$$

If n is not an integer the value of the definite integral is much more complicated. It is obviously a function of n. $\int_0^\infty e^{-x}x^{n-1}dx$ is generally represented by $\Gamma(n)$, and it has been carefully studied under the name of the *Gamma Function.*

In every case we have by (1),

$$\Gamma(n+1)=n\Gamma(n).$$

$\int_0^1 (\log x)^n dx$ can be reduced to $(-1)^n \int_0^\infty e^{-z} z^n dz$, and therefore $= \pm\,\Gamma(n+1)$.

83. We have seen that if fx becomes infinite for a value of x between a and b,

$$\int_a^b fx.dx \text{ is not equal to } \left[\int fx.dx\right]_{x=b} - \left[\int fx.dx\right]_{x=a}.$$

Suppose that $fx=\infty$ when $x=c$, $\int_a^{c-\varepsilon} fx.dx$ and $\int_{c+\varepsilon}^b fx.dx$ can be found without difficulty.

$\underset{\epsilon \doteq 0}{\text{limit}}\left[\int_a^{c-\varepsilon} fx.dx + \int_{c+\varepsilon}^b fx.dx\right]$ is called the *principal value* of $\int_a^b fx.d\phi$, and is often finite and determinate.

For example; let us find $\int_a^b \frac{dx}{c-x}$.

$$\frac{1}{c-x}=\infty \text{ when } x=c.$$

$$\int_a^{c-\varepsilon} \frac{dx}{c-x} = \log\frac{c-a}{\epsilon},$$

$$\int_{c+\varepsilon}^b \frac{dx}{c-x} = -\log\frac{b-c}{\epsilon},$$

and the *principal value* of $\int_a^b \frac{dx}{c-x} = \log\frac{c-a}{b-c}$.

EXAMPLE.

Find the *principal value* of $\int_{-x_0}^{X} \frac{dx}{x^3}$. *Ans.* $\frac{1}{2}\left(\frac{1}{x_0^2}-\frac{1}{X^2}\right)$.

84. We have seen, Art. 51, that a definite integral is a function of the *limits of integration*, and not of the variable explicitly

appearing in the expression integrated. Let us consider the possibility of differentiating a definite integral.

Required $D_\alpha\int_a^b f(x,\alpha)dx$ where α is independent of x, and a and b do not depend upon α. Of course our derivative is a *partial derivative.*

Let $\quad u=\int_a^b f(x,\alpha)dx,$

$$D_\alpha u=\underset{\Delta\alpha \doteq 0}{\text{limit}}\left[\frac{\int_a^b f(x,\alpha+\Delta\alpha)dx-\int_a^b f(x,\alpha)dx}{\Delta\alpha}\right]$$

$$=\underset{\Delta\alpha \doteq 0}{\text{limit}}\left[\int_a^b\frac{f(x,\alpha+\Delta\alpha)-f(x,\alpha)}{\Delta\alpha}dx\right],$$

$$D_\alpha\int_a^b f(x,\alpha)dx=\int_a^b D_\alpha f(x,\alpha)dx,$$

and we find that we can differentiate under the sign of integration.

The truth of the converse of the last proposition can be easily established.

$$\int\left[\int_a^b f(x,\alpha)dx\right]d\alpha=\int_a^b\left[\int f(x,\alpha)d\alpha\right]dx$$

or even $\quad \int_{\alpha_0}^{\alpha}\left[\int_a^b f(x,\alpha)dx\right]d\alpha=\int_a^b\left[\int_{\alpha_0}^{\alpha} f(x,\alpha)d\alpha\right]dx$ if a, b, α_0, and α are entirely independent.

A skilful use of the principles of the present Article will often enable us to obtain new definite integrals.

For example; $\quad \int_0^\infty e^{-ax}dx=\frac{1}{a}.$

Differentiate both members with respect to a;

$$\int_0^\infty(-xe^{-ax}dx)=-\frac{1}{a^2}\quad\text{or}\quad\int_0^\infty(xe^{-ax}dx=\frac{1}{a^2}.$$

Differentiate again;

$$\int_0^\infty x^2e^{-ax}dx=\frac{2\,!}{a^3}.$$

Differentiating n times;

$$\int_0^\infty x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}.$$

Let us consider $\displaystyle\int_0^\infty \frac{dx}{x^2+a} = \frac{\pi}{2}\cdot\frac{1}{a^{\frac{1}{2}}}.$

Differentiate n times with respect to a;

$$\int_0^\infty \frac{dx}{(x^2+a)^{n+1}} = \frac{\pi}{2}\cdot\frac{1.3.5\cdots\cdots(2n-1)}{2.4.6\cdots\cdots 2n}\cdot\frac{1}{a^{n+\frac{1}{2}}}.$$

85. By differentiating under the sign of integration a complicated form may sometimes be reduced to a simpler one.

Required $\displaystyle\int_0^\infty \frac{e^{-ax}\sin mx.dx}{x}.$

Let $\displaystyle u = \int_0^\infty \frac{e^{-ax}\sin mx}{x}dx,$

$$D_m u = \int_0^\infty e^{-ax}\cos mx.dx = \frac{a}{a^2+m^2}, \qquad \text{by Art. 80, Ex. 7.}$$

Hence $\displaystyle u = \int\frac{a}{a^2+m^2}dm = a\int\frac{dm}{a^2+m^2} = \tan^{-1}\frac{m}{a} + C.$

Since, when $m=0$, $\dfrac{e^{-ax}\sin mx}{x}dx$ is constantly zero, $\displaystyle\int_0^\infty\frac{e^{-ax}\sin mx}{x}dx = 0$ when $m=0$, and therefore $C=0$, and we have

$$\int_0^\infty \frac{e^{-ax}\sin mx}{x}dx = \tan^{-1}\frac{m}{a}.$$

EXAMPLE.

Obtain from $\displaystyle\int_0^1 x^n\,dx = \frac{1}{n+1}$

$$\int_0^1 x^n(\log x)^k\,dx = (-1)^k\frac{k!}{(n+1)^{k+1}}.$$

86. We know that $\int_0^1 x^{\alpha-1}dx = \frac{1}{\alpha}$.

Integrate both members

$$\int_0^1 \left(\int_{\alpha_0}^{\alpha_1} x^{\alpha-1}\, d\alpha\right) dx = \int_{\alpha_0}^{\alpha_1} \frac{d\alpha}{\alpha},$$

$$\int_0^1 \frac{x^{\alpha_1-1} - x^{\alpha_0-1}}{\log x}\, dx = \log\frac{\alpha_1}{\alpha_0}.$$

EXAMPLES.

(1) From $\int_0^\infty e^{-\alpha x}\cos mx.dx = \frac{\alpha}{\alpha^2+m^2}$ obtain

$$\int_0^\infty \frac{e^{-\alpha_1 x} - e^{-\alpha_0 x}}{x}\cos mx.dx = \tfrac{1}{2}\log\left(\frac{\alpha_0^2+m^2}{\alpha_1^2+m^2}\right).$$

(2) From $\int_0^\infty e^{-\alpha x}\sin mx.dx = \frac{m}{\alpha^2+m^2}$ obtain

$$\int_0^\infty \frac{e^{-\alpha_0 x} - e^{-\alpha_1 x}}{x}\sin mx.dx = \tan^{-1}\frac{\alpha_1}{m} - \tan^{-1}\frac{\alpha_0}{m}.$$

87. If in $\int_a^b f(x,\alpha)dx$ a and b are functions of α, our formula for $\frac{d}{d\alpha}\int_a^b f(x,\alpha)dx$ becomes more complicated.

Let $$\int f(x,\alpha)dx = F(x,\alpha),$$

then $$u = \int_a^b f(x,\alpha)dx = F(b,\alpha) - F(a,\alpha),$$

$$\frac{du}{d\alpha} = \frac{d}{d\alpha}F(b,\alpha) - \frac{d}{d\alpha}F(a,\alpha);$$

but as b and a are functions of α,

$$\frac{d}{d\alpha}F(b,\alpha) = D_bF(b,\alpha)\frac{db}{d\alpha} + D_\alpha F(b,\alpha),$$

$$\frac{d}{d\alpha}F(a,\alpha) = D_aF(a,\alpha)\frac{da}{d\alpha} + D_\alpha F(a,\alpha),$$ by I. Art. 200.

$$D_b F(b,a) = f(b,a),$$

$$D_a F(a,a) = f(a,a);$$

$$\frac{du}{da} = D_a[F(b,a) - Fa,a] + f(b,a)\frac{db}{da} - f(a,a)\frac{da}{da}$$

$$= \int_a^b D_a f(x,a)\,dx) + f(b,a)\frac{db}{da} - f(a,a)\frac{da}{da}. \qquad [1]$$

EXAMPLE.

Confirm this formula by finding $\frac{d}{dy}\int_0^{xy}\sin(x+y)\,dx$.

88. In our treatment of definite integrals we have supposed that our *limits of integration* were real, and that the increment dx of the independent variable was always real.

Definite integrals taken between imaginary limits, and formed by giving the variable imaginary increments, have been made the subject of careful study, and they are found very useful in connection with the subject of *Elliptic Integrals* (Art. 72), or, as they are sometimes called, *Doubly Periodic Functions*.

CHAPTER IX.

LENGTHS OF CURVES.

89. If we use rectangular coördinates, we have seen (I. Art. 27) that

$$\tan\tau = \frac{dy}{dx}, \qquad [1]$$

and (I. Arts. 52 and 181) that

$$ds^2 = dx^2 + dy^2. \qquad [2]$$

From these we get

$$\sin\tau = \frac{dy}{ds}, \qquad [3]$$

$$\cos\tau = \frac{dx}{ds}, \qquad [4]$$

by the aid of a little elementary Trigonometry.

These formulas are of great importance in dealing with all properties of curves that concern in any way the lengths of arcs.

We have already considered the use of [2] in the first volume of the Calculus, and we have worked several examples by its aid in rectification of curves. Before going on to more of the same sort we shall find it worth while to obtain the equations of two very interesting transcendental curves, the *catenary* and the *tractrix*.

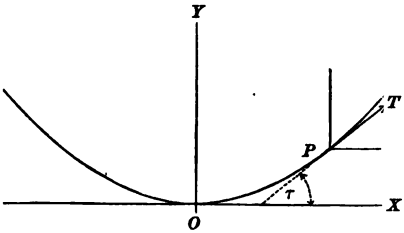
The Catenary.

90. The common *catenary* is the curve in which a uniform heavy flexible string hangs when its ends are supported.

As the string is flexible, the only force exerted by one portion of the string on an adjacent portion is a pull along the string, which we shall call the tension of the string, and shall represent by T. T of course has different values at different points of the string, and is some function of the coördinates of the point in question.

The tension at any point has to support the weight of the portion of the string below the point, and a certain amount of side pull, due to the fact that the string would hang vertically were it not that its ends are forcibly held apart.

Let the origin be taken at the lowest point of the curve, and suppose the string fastened at that point.



Let s be the arc OP, P being any point of the string. As the string is uniform, the weight of OP is proportional to its length; we shall call this weight ms.

This weight acts vertically downward, and must be balanced by the vertical effect of T, which, by I. Art. 112, is $T \sin \tau$.

Hence $$T \sin \tau = ms. \tag{1}$$

As there is no *external* horizontal force acting, the horizontal effect of the tension at one end of any portion of the string must be the same as the horizontal effect at the other end. In other words, $$T \cos \tau = c \tag{2}$$

where c is a constant. Dividing (1) by (2) we get

$$s = \frac{c}{m} \tan \tau,$$

or $$s = a \tan \tau, \tag{3}$$

where a is some constant. From this we want to get an equation in terms of x and y.

$$\tan \tau = \sqrt{\sec^2 \tau - 1} = \sqrt{\frac{ds^2}{dx^2} - 1};$$

hence $$s^2 = a^2 \left(\frac{ds^2}{dx^2} - 1 \right),$$

or $$a^2 ds^2 = (a^2 + s^2) dx^2,$$

and $$\frac{ads}{(a^2 + s^2)^{\frac{1}{2}}} = dx.$$ Integrate both members.

$$a\log(s+\sqrt{a^2+s^2})=x+C;$$

when $x=0$, $s=0$,

hence $$C=\log a,$$

and $$\log(s+\sqrt{a^2+s^2})=\frac{x}{a}+\log a,$$

$$s+\sqrt{a^2+s^2}=ae^{\frac{x}{a}},$$

$$\sqrt{a^2+s^2}=ae^{\frac{x}{a}}-s,$$

$$a^2=a^2e^{2\frac{x}{a}}-2ae^{\frac{x}{a}}s,$$

$$s=\frac{a}{2}(e^{\frac{x}{a}}-e^{-\frac{x}{a}})=a\tan\tau \quad \text{by (3).}$$

Hence $$a\frac{dy}{dx}=\frac{a}{2}(e^{\frac{x}{a}}-e^{-\frac{x}{a}}),$$

and $$y=\frac{a}{2}(e^{\frac{x}{a}}+e^{-\frac{x}{a}})+C.$$

If we change our axes, taking the origin at a point a units below the lowest point of the curve, $y=a$ when $x=0$, and therefore $C=0$, and we get, as the equation of the catenary,

$$y=\frac{a}{2}(e^{\frac{x}{a}}+e^{-\frac{x}{a}}). \tag{4}$$

EXAMPLE.

Find the curve in which the cables of a suspension-bridge must hang. *Ans.* A parabola.

The Tractrix.

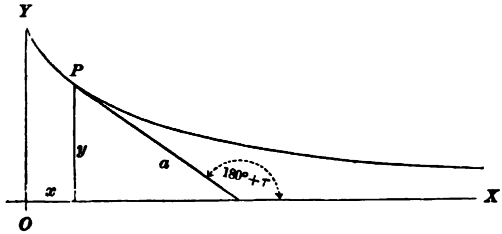
91. If two particles are attached to a string, and rest on a rough horizontal plane, and one, starting with the string stretched, moves in a straight line at right angles with the initial position of the string, dragging the other particle after it, the path of the second particle is called the *tractrix.*

Take as the axis of X the path of the first particle, and as the axis of Y the initial position of the string, and let a be

the length of the string. From the nature of the curve the string is always a tangent, and we shall have for any point P

$$\frac{y}{a} = -\sin\tau, \tag{1}$$

for τ lying in the fourth quadrant has a negative sine.



$$\frac{y^2}{a^2} = \sin^2\tau = \frac{dy^2}{ds^2} = \frac{dy^2}{dx^2 + dy^2};$$

hence $$y^2(dx^2 + dy^2) = a^2 dy^2,$$

$$y^2 dx^2 = (a^2 - y^2) dy^2,$$

and $$dx = \pm \frac{(a^2 - y^2)^{\frac{1}{2}} dy}{y}$$

is the differential equation of the *tractrix*.

On the right-hand half of the curve τ is in the fourth quadrant, $\frac{dy}{dx}$ or $\tan\tau$ is negative, and we shall write the equation

$$dx = -\frac{(a^2 - y^2)^{\frac{1}{2}} dy}{y}. \tag{2}$$

If we allow the radical to be ambiguous in sign we shall get also the curve that would be described if the first particle went to the left instead of to the right. The tractrix curve, generally considered, includes these two portions.

Integrating both members of [2], and determining the arbitrary constant, we get

$$x = -\sqrt{a^2 - y^2} + a \log \frac{a + \sqrt{a^2 - y^2}}{y} \tag{3}$$

as the equation of the *tractrix*.

EXAMPLES.

(1) Show by Art. 91 (1) that in the tractrix $s = a \log \frac{a}{y}$ if s is measured from the starting-point.

(2) Find the evolute of the tractrix. (I. Art. 93.)

Rectification of Curves.

92. In finding the length of an arc of a given curve we can regard it as the limit of the sum of the differentials of the arc, and express it by a definite integral.

We shall have $$s = \int_{x=x_0}^{x=x_1} \sqrt{dx^2 + dy^2}.$$

Of course in using this formula we must express $\sqrt{dx^2 + dy^2}$ in terms of x only, or of y only, or of some single variable on which x and y depend, before we can integrate.

For example; let us find the length of an arc of the circle

$$x^2 + y^2 = a^2.$$

$$2x.dx + 2y.dy = 0,$$

$$dy = -\frac{x.dx}{y},$$

$$dx^2 + dy^2 = \frac{x^2 + y^2}{y^2} dx^2 = \frac{a^2}{y^2} dx^2 = \frac{a^2}{a^2 - x^2} dx^2,$$

$$s = a \int_{x_0}^{x_1} \frac{dx}{\sqrt{a^2 - x^2}} = a \left(\sin^{-1} \frac{x_1}{a} - \sin^{-1} \frac{x_0}{a} \right).$$

The length of a quadrant $= a \int_0^a \frac{dx}{\sqrt{a^2 - x^2}} = \frac{\pi a}{2}$;

$\therefore$ the length of a circumference $= 2\pi a$.

Length of Arc of Cycloid.

93. For the cycloid we have

$$\left.\begin{aligned} x &= a\theta - a\sin\theta \\ y &= a \quad - a\cos\theta \end{aligned}\right\}. \qquad \text{(I. Art. 99.)}$$

$$dx = a(1-\cos\theta)\,d\theta = y\,d\theta,$$

$$\theta = \text{vers}^{-1}\frac{y}{a},$$

$$d\theta = \frac{1}{a}\cdot\frac{dy}{\sqrt{2\frac{y}{a}-\frac{y^2}{a^2}}} = \frac{dy}{\sqrt{2ay-y^2}},$$

$$dx = \frac{y\,dy}{\sqrt{2ay-y^2}},$$

$$ds^2 = dx^2 + dy^2 = \frac{2ay\,dy^2}{2ay-y^2} = \frac{2a\,dy^2}{2a-y},$$

$$ds = \sqrt{2a}\,.\frac{dy}{\sqrt{2a-y}},$$

$$s = \sqrt{2a}\int_{y_0}^{y_1}\frac{dy}{\sqrt{2a-y}} = 2\sqrt{2a}\left(\sqrt{2a-y_0}-\sqrt{2a-y_1}\right).$$

If the arc is measured from the cusp, $y_0 = 0$,

$$s = 4a - 2\sqrt{2a}\sqrt{2a-y_1}. \qquad [1]$$

If the arc is measured to the highest point, $y_1 = 2a$,

$$s = 4a.$$

The whole arch $= 8a$.

EXAMPLE.

Taking the origin at the vertex, and taking the direction downward as the positive direction for y, the equations become

$$\left.\begin{aligned} x &= a\theta + a\sin\theta \\ y &= \quad a - a\cos\theta \end{aligned}\right\}. \qquad \text{(I. Art. 100.)}$$

Show that $s = 2\sqrt{2ay}$ when the arc is measured from the summit of the curve.

94. We can rectify the cycloid without eliminating θ.

$$\left.\begin{array}{l} x = a\theta - a\sin\theta \\ y = \phantom{a\theta -{}} a - a\cos\theta \end{array}\right\},$$

$$dx = a(1-\cos\theta)\,d\theta,$$

$$dy = a\sin\theta.d\theta,$$

$$dx^2 + dy^2 = 2a^2 d\theta^2(1-\cos\theta),$$

and
$$s = a\sqrt{2}\int_{\theta_0}^{\theta_1}(1-\cos\theta)^{\frac{1}{2}}d\theta,$$

$$s = a\sqrt{2}\int_{\theta_0}^{\theta_1}\left[2\sin^2\frac{\theta}{2}\right]^{\frac{1}{2}}d\theta = 4a\int_{\theta_0}^{\theta_1}\sin\frac{\theta}{2}\,d\frac{\theta}{2} = 4a\left(\cos\frac{\theta_0}{2} - \cos\frac{\theta_1}{2}\right).$$

If $\theta_0 = 0$ and $\theta_1 = 2\pi$, we get $s = 8a$ as the whole curve.

95. Let us find the *length of an arch of the epicycloid.*

$$\left.\begin{array}{l} x = (a+b)\cos\theta - b\cos\dfrac{(a+b)}{b}\theta \\ y = (a+b)\sin\theta - b\sin\dfrac{a+b}{b}\theta \end{array}\right\}, \text{(I. Art.109[1].)}$$

$$dx = \left[-(a+b)\sin\theta + (a+b)\sin\frac{a+b}{b}\theta\right]d\theta,$$

$$dy = \left[(a+b)\cos\theta - (a+b)\cos\frac{a+b}{b}\theta\right]d\theta.$$

$$ds^2 = (a+b)^2 d\theta^2\left[2 - 2\left(\cos\frac{a+b}{b}\theta\cos\theta + \sin\frac{a+b}{b}\theta\sin\theta\right)\right]$$

$$= 2(a+b)^2 d\theta^2\left(1 - \cos\frac{a}{b}\theta\right).$$

$$s = (a+b)\sqrt{2}\int_{\theta_0}^{\theta_1}\left(1 - \cos\frac{a}{b}\theta\right)^{\frac{1}{2}}d\theta,$$

$$s = \frac{4b(a+b)}{a}\left[\cos\frac{a}{2b}\theta_0 - \cos\frac{a}{2b}\theta_1\right]. \qquad [1]$$

To get a complete arch we must let $\theta_0 = 0$ and $\theta_1 = \dfrac{2b}{a}\pi$. Hence, for a whole arch,

$$s = \frac{8b(a+b)}{a}.$$

EXAMPLES.

(1) Find the length of an arch of a hypocycloid.

(2) Find an arc of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, and see whether it agrees with the result of Ex. (1); see I. Art. 109, Ex. 2.

96. Let us attempt to find the *length of an arc of the ellipse.*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

$$\frac{2x.dx}{a^2} + \frac{2y.dy}{b^2} = 0,$$

$$dy = -\frac{b^2 x}{a^2 y} dx,$$

$$ds^2 = \frac{b^4 x^2 + a^4 y^2}{a^4 y^2} dx^2 = \left[1 + \frac{b^2 x^2}{a^2(a^2 - x^2)}\right] dx^2,$$

$$s = \int_{x_0}^{x_1} \left[1 + \frac{b^2 x^2}{a^2(a^2 - x^2)}\right]^{\frac{1}{2}} dx.$$

This function cannot be integrated directly; $\left[1 + \frac{b^2 x^2}{a^2(a^2 - x^2)}\right]^{\frac{1}{2}}$ can be expanded by the Binomial Theorem, and the terms can be integrated separately, and we shall have the length of the arc expressed by a complicated series.

A more convenient way of dealing with the problem is to use an auxiliary angle. Instead of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ we can use the pair of equations

$$\left.\begin{aligned} x &= a \sin\phi \\ y &= b\cos\phi \end{aligned}\right\}, \qquad \text{(I. Art. 150)},$$

$$dx = a \cos\phi.d\phi,$$

$$dy = -b \sin\phi.d\phi,$$

$$ds^2 = (a^2 \cos^2\phi + b^2 \sin^2\phi) d\phi^2 = [a^2 - (a^2 - b^2)\sin^2\phi] d\phi^2$$

$$= a^2\left(1 - \frac{a^2 - b^2}{a^2}\sin^2\phi\right) d\phi^2 = a^2(1 - e^2 \sin^2\phi) d\phi^2,$$

where e is the eccentricity of the ellipse.

$$s = a\int_{\phi_0}^{\phi_1}(1 - e^2\sin^2\phi)^{\frac{1}{2}}d\phi$$

$$= a\int_{\phi_0}^{\phi_1}\left[1 - \frac{1}{2}e^2\sin^2\phi - \frac{1}{2}\cdot\frac{1}{4}e^4\sin^4\phi - \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{6}e^6\sin^6\phi \ldots\ldots\right]d\phi.$$

This is one of the famous *Elliptic Integrals.*

EXAMPLE.

Show that the length of the hyperbolic arc is

$$s = b\int_{\phi_0}^{\phi_1}\left[1 + \frac{a^2e^2}{b^2}\mathrm{Sh}^2\phi\right]^{\frac{1}{2}}d\phi.$$

Polar Formulae.

97. If we use polar coördinates we have

$$ds = \sqrt{dr^2 + r^2 d\phi^2}, \quad \text{(I. Art. 207, Ex. 2)},$$

$$\tan\epsilon = \frac{rd\phi}{dr}, \quad \text{(I. Art. 207)}.$$

From these we get, by Trigonometry,

$$\sin\epsilon = \frac{rd\phi}{ds}, \tag{3}$$

$$\cos\epsilon = \frac{dr}{ds}. \tag{4}$$

98. Let us find the equation of the curve which crosses all its radii vectores at the same angle. Here

$$\tan\epsilon = a, \quad \text{a constant},$$

$$\frac{rd\phi}{dr} = a,$$

$$\frac{adr}{r} = d\phi,$$

$$a\log r = \phi + C,$$

$$r = e^{\frac{\phi}{a} + \frac{c}{a}} = e^{\frac{c}{a}} e^{\frac{\phi}{a}},$$

$$r = be^{\frac{\phi}{a}} \tag{1}$$

where b is some constant depending upon the position of the origin. This curve is known as the *Logarithmic* or *Equiangular* Spiral.

99. To rectify the *Logarithmic Spiral*. We have, from 98 (1),

$$\frac{\phi}{a} = \log\frac{r}{b};$$

$$d\phi = a\frac{dr}{r},$$

$$rd\phi = adr,$$

$$ds^2 = dr^2 + r^2 d\phi^2 = (1 + a^2) dr^2;$$

$$s = \int_{r_0}^{r_1} (1 + a^2)^{\frac{1}{2}} dr = (1 + a^2)^{\frac{1}{2}} (r_1 - r_0).$$

EXAMPLES.

(1) Find the length of an arc of the parabola from its polar equation

$$r = \frac{m}{1 + \cos\phi}.$$

(2) Find the length of an arc of the Spiral of Archimedes

$$r = a\phi.$$

100. To rectify the *Cardioide*. We have

$$r = 2a(1 - \cos\phi), \qquad \text{(I. Art. 109, Ex. 1)},$$

$$dr = 2a\sin\phi . d\phi,$$

$$ds^2 = 4a^2 \sin^2\phi . d\phi^2 + 4a^2 (1 - \cos\phi)^2 d\phi^2$$

$$= 8a^2 d\phi^2 (1 - \cos\phi),$$

$$s = 2\sqrt{2} . a \int_{\phi_0}^{\phi_1} (1 - \cos\phi)^{\frac{1}{2}} d\phi = 8a\left[\cos\frac{\phi_0}{2} - \cos\frac{\phi_1}{2}\right]$$

$= 16a$ for the whole perimeter.

Involutes.

101. If we can express the length of the arc of a given curve, measured from a fixed point, in terms of the coördinates of its variable extremity, we can find the equation of the *involute* of the curve.

We have found the equations of the evolute of $y = fx$ in the form

$$\left.\begin{aligned} x' &= x - \rho\cos\nu \\ y' &= y - \rho\sin\nu \end{aligned}\right\}, \qquad \text{(I. Art. 91).}$$

We have proved that $\quad \tan\nu = \tan\tau', \qquad$ (I. Art. 95),

and that

$$\frac{ds'}{d\rho} = 1, \qquad \text{(I. Art. 96)};$$

$$\left.\begin{aligned} \sin\tau' &= \frac{dy'}{ds'} \\ \cos\tau' &= \frac{dx'}{ds'} \end{aligned}\right\}, \qquad \text{(Art. 89).}$$

Since $\quad \tan\nu = \tan\tau', \quad \nu = \tau' \;$ or $\; \nu = 180° + \tau'$.

As normal and radius of curvature have opposite directions, we shall consider $\nu = 180° + \tau'$.

Then $\quad \sin\nu = -\sin\tau' \;$ and $\; \cos\nu = -\cos\tau'$.

Hence

$$x' = x + \rho\frac{dx'}{ds'}, \tag{1}$$

$$y' = y + \rho\frac{dy'}{ds'}. \tag{2}$$

Since

$$d\rho = ds',$$

$$\rho = s' + l \tag{3}$$

where l is an arbitrary constant. x and y being the coördinates of any point of the *involute*, it is only necessary to eliminate x', y', and ρ by combining equations (1), (2), and (3) with the equation of the evolute.

As we are supposed to start with the equation of the evolute and work towards the equation of the involute, it will be more natural to accent the letters belonging to the latter curve instead

of those going with the former; and our equations may be written

$$x = x' + \rho' \frac{dx}{ds}; \tag{4}$$

$$y = y' + \rho' \frac{dy}{ds}; \tag{5}$$

$$\rho' = s + l. \tag{6}$$

To test our method, let us find the involute of the curve

$$y^2 = \frac{8}{27m}(x - m)^3, \tag{7}$$

letting $\rho' = s + m$. We must first find s.

$$2y\,dy = \frac{8}{9m}(x - m)^2 dx,$$

$$dy = \frac{4}{9m}\frac{(x - m)^2}{y} dx,$$

$$ds^2 = \frac{2x + m}{3m} dx^2,$$

$$s = \frac{1}{\sqrt{3m}}\int_m^x (2x + m)^{\frac{1}{2}} dx = \frac{1}{3\sqrt{3m}}(2x + m)^{\frac{3}{2}} - m,$$

$$\rho' = s + m = \frac{1}{3\sqrt{3m}}(2x + m)^{\frac{3}{2}},$$

$$x = x' + \frac{2x + m}{3},$$

$$y = y' + \frac{4}{27m}\frac{(2x + m)(x - m)^2}{y},$$

$$x' = \frac{x - m}{3},$$

$$y' = -\frac{4}{9y}(x - m)^2 = -\frac{4x'^2}{y},$$

$$x = 3x' + m,$$

$$y = -\frac{4x'^2}{y'}.$$

Substituting in (7) the values of x and y just obtained, we have

$$y'^2 = 2mx'$$

as the equations of the required involute.

EXAMPLE.

Find the involute of $ay^2 = x^3$.

102. The involute of the cycloid is easily found. Take equations I. Art. 100 (C),

$$\left.\begin{aligned} x &= a\theta + a\sin\theta \\ y &= -a + a\cos\theta \end{aligned}\right\}.$$

Let $\rho' = s,$

$$dx = a(1+\cos\theta)d\theta \quad = 2a\cos^2\frac{\theta}{2}d\theta,$$

$$dy = -a\sin\theta . d\theta \quad = -2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}d\theta,$$

$$ds^2 = 2a^2 d\theta^2(1+\cos\theta) = 4a^2 d\theta^2 \cos^2\frac{\theta}{2},$$

$$s = 2a\int_0^\theta \cos\frac{\theta}{2}d\theta \quad = 4a\sin\frac{\theta}{2},$$

$$x = x' + 4a\sin\frac{\theta}{2}\cos\frac{\theta}{2} = x' + 2a\sin\theta,$$

$$y = y' + 4a\sin^2\frac{\theta}{2} \quad = y' - 2a(1-\cos\theta),$$

$$\left.\begin{aligned} x' &= a\theta - a\sin\theta \\ y' &= a - a\cos\theta \end{aligned}\right\}$$

a cycloid with its cusp at the summit of the given cycloid.

EXAMPLE.

From the equations of a circle

$$\left.\begin{aligned} x &= a\cos\phi \\ y &= a\sin\phi \end{aligned}\right\}$$

obtain the equations of the involute of the circle.

Ans. $$\left.\begin{aligned} x' &= a(\cos\phi + \phi\sin\phi) \\ y' &= a(\sin\phi - \phi\cos\phi) \end{aligned}\right\}.$$

Intrinsic Equation of a Curve.

103. An equation connecting the length of the arc, measured from a fixed point of any curve to a variable point, with the angle between the tangent at the fixed point and the tangent at the variable point, is the *intrinsic equation* of the curve. If the fixed point is the origin and the fixed tangent the axis of X, the variables in the *intrinsic* equation are s and τ.

We have already such an equation for the catenary

$$s = a \tan \tau, \qquad \text{Art. 90 (3)}, \qquad [1]$$

the origin being the lowest point of the curve.

The intrinsic equation of a circle is obviously

$$s = a\tau, \qquad [2]$$

whatever origin we may take.

The intrinsic equation of the tractrix is easily obtained. We have

$$y = -a \sin \tau, \qquad \text{Art. 91 (1)},$$

and

$$s = a \log \frac{a}{y}; \qquad \text{Art. 91, Ex. 1},$$

hence

$$s = a \log(-\csc \tau)$$

where τ is measured from the axis of X, and s is measured from the point where the curve crosses the axis of Y. As the curve is tangent to the axis of Y, we must replace τ by $\tau - 90°$, and we get

$$s = a \log \sec \tau \qquad [3]$$

as the intrinsic equation of the tractrix.

EXAMPLE.

Show that the intrinsic equation of an inverted cycloid, when the vertex is origin, is

$$s = 4a \sin \tau; \qquad (1)$$

when the cusp is origin, is

$$s = 4a(1 - \cos \tau). \qquad (2)$$

104. To find the intrinsic equation of the epicycloid we can use the results obtained in Art. 95.

$$dx = (a+b)\left(\sin\frac{a+b}{b}\theta - \sin\theta\right)d\theta = 2(a+b)\cos\frac{a+2b}{2b}\theta\sin\frac{a}{2b}\theta . d\theta,$$

$$dy = (a+b)\left(\cos\theta - \cos\frac{a+b}{b}\theta\right)d\theta = 2(a+b)\sin\frac{a+2b}{2b}\theta\sin\frac{a}{2b}\theta . d\theta,$$

by the formulas of Trigonometry;

$$\sin\alpha - \sin\beta = 2\cos\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\alpha-\beta),$$

$$\cos\beta - \cos\alpha = 2\sin\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\alpha-\beta);$$

$$\tan\tau = \frac{dy}{dx} = \tan\frac{a+2b}{2b}\theta,$$

hence $$\tau = \frac{a+2b}{2b}\theta;$$

$$s = \frac{4b(a+b)}{a}\left(1 - \cos\frac{a}{2b}\theta\right) \text{ by Art. 95 [1];}$$

therefore $$s = \frac{4b(a+b)}{a}\left(1 - \cos\frac{a}{a+2b}\tau\right) \qquad [1]$$

is the intrinsic equation of the epicycloid, with the *cusp* as origin.

If we take the origin at a vertex instead of at a cusp

$$s = \frac{4b(a+b)}{a} + s',$$

$$\tau = \frac{\pi(a+2b)}{2a} + \tau';$$

hence $$s' = \frac{4b(a+b)}{a}\sin\frac{a}{a+2b}\tau';$$

or $$s = \frac{4b(a+b)}{a}\sin\frac{a}{a+2b}\tau$$

is the intrinsic equation of an epicycloid referred to a vertex.

EXAMPLE.

Obtain the intrinsic equation of the hypocycloid in the forms

$$s=\frac{4b(a-b)}{a}\left(1-\cos\frac{a}{a-2b}\tau\right), \tag{1}$$

$$s=\frac{4b(a-b)}{a}\sin\frac{a}{a-2b}\tau. \tag{2}$$

105. The intrinsic equation of the Logarithmic Spiral is found without difficulty.

We have $$r=be^{\frac{\phi}{a}}, \qquad \text{(Art. 98)},$$

and $$s=\sqrt{1+a^2}(r_1-r_0). \qquad \text{(Art. 99)}.$$

If we measure the arc from the point where the spiral crosses the initial line, $r_0=b$, and we have

$$s=b\sqrt{1+a^2}(e^{\frac{\phi}{a}}-1).$$

In polar coördinates $\tau=\phi+\epsilon$, and in this case $\epsilon=\tan^{-1}a$; if we measure our angle from the tangent at the beginning of the arc we must subtract ϵ from the value just given, and we have

$$s=b(\sqrt{1+a^2})(e^{\frac{\tau}{a}}-1);$$

or, more briefly, $s=k(c^\tau-1)$, k and c being constants.

106. If we wish to get the intrinsic equation of a curve directly from the equation in rectangular coördinates, the following method will serve:

Let the axis of X be tangent to the curve at the point we take as origin.

$$\tan\tau=\frac{dy}{dx}; \tag{1}$$

and as the equation of the curve enables us to express y in terms of x, (1) will give us x in terms of τ, say $x=F\tau$;

then $\qquad dx = F'\tau . d\tau,$ $\qquad$ divide by ds;

$$\frac{dx}{ds} = F'\tau \frac{d\tau}{ds}, \quad \text{but} \quad \frac{dx}{ds} = \cos\tau;$$

hence $\qquad ds = \sec\tau F'\tau . d\tau. \qquad (2)$

Integrating both members we shall have the required intrinsic equation.

For example, let us take $x^2 = 2my$, which is tangent to the axis of X at the origin.

$$2x dx = 2m dy,$$

$$\frac{dy}{dx} = \tan\tau = \frac{x}{m},$$

$$dx = m\sec^2\tau . d\tau,$$

$$\frac{dx}{ds} = \cos\tau = m\sec^2\tau \frac{d\tau}{ds},$$

$$ds = m\sec^3\tau . d\tau, \qquad (1)$$

$$s = m\int\frac{d\tau}{\cos^3\tau} = \frac{m}{2}\left[\frac{\sin\tau}{\cos^2\tau} + \log\tan\left(\frac{\pi}{4} + \frac{\tau}{2}\right)\right] + C,$$

$$s = 0 \text{ when } \tau = 0; \qquad \therefore C = 0;$$

$$s = \frac{m}{2}\left[\frac{\sin\tau}{\cos^2\tau} + \log\tan\left(\frac{\pi}{4} + \frac{\tau}{2}\right)\right]. \qquad (2)$$

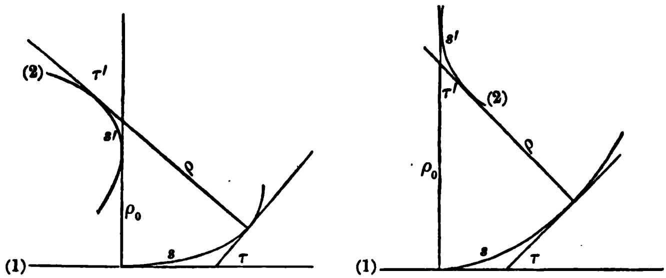
EXAMPLES.

(1) Devise a method when the curve is tangent to the axis of Y, and apply it to $y^2 = 2mx$.

(2) Obtain the intrinsic equation of $y^2 = \dfrac{8}{27m}(x - m)^3$.

(3) Obtain the intrinsic equation of the involute of a circle. (Art. 102, Ex.)

107. The *evolute* or the *involute* of a curve is easily found from its intrinsic equation.



If the curvature of the given curve decreases as we pass along the curve, ρ increases, and

$$s' = \rho - \rho_0. \qquad \text{(I. Art. 96).}$$

If the curvature increases, ρ decreases, and

$$s' = \rho_0 - \rho.$$

Hence always

$$s' = \pm(\rho - \rho_0); \qquad [1]$$

$$\rho = \frac{ds}{d\tau}, \qquad \text{(I. Arts. 86 and 90).}$$

We see from the figure that $\tau' = \tau$.

Hence

$$s' = \pm\left[\left(\frac{ds}{d\tau}\right)_{\tau=\tau'} - \left(\frac{ds}{d\tau}\right)_{\tau=0}\right],$$

or, as we shall write it for brevity,

$$s = \pm\left.\frac{ds}{d\tau}\right|_0^\tau. \qquad [2]$$

108. The evolute of the *tractrix* $s = a \log\sec\tau$ is

$$s = a\left.\frac{d\log\sec\tau}{d\tau}\right|_0^\tau = a\tan\tau, \quad \text{the catenary.}$$

The evolute of the circle $s = a\tau$ is

$$s = a\left.\frac{d\tau}{d\tau}\right|_0^\tau = 0, \quad \text{a point.}$$

The evolute of the cycloid $s = 4a(1 - \cos\tau)$ is

$$s = 4a \left.\frac{d(1 - \cos\tau)}{d\tau}\right|_0^\tau = 4a\sin\tau,$$

an equal cycloid, with its vertex at the origin.

EXAMPLES.

(1) Prove that the evolute of the logarithmic spiral is an equal logarithmic spiral.

(2) Find the evolute of a parabola.

(3) Find the evolute of the catenary.

109. The evolute of an epicycloid is a similar epicycloid, with each vertex at a cusp of the given curve.

Take the equation

$$s = \frac{4b(a+b)}{a}\left(1 - \cos\frac{a}{a+2b}\tau\right). \qquad \text{Art. 104 [1].}$$

For the evolute,

$$s = \frac{4b(a+b)}{a} \left.\frac{d\left(1 - \cos\frac{a}{a+2b}\tau\right)}{d\tau}\right|_0^\tau,$$

$$s = \frac{4b(a+b)}{a+2b}\sin\frac{a}{a+2b}\tau. \qquad [1]$$

The form of [1] is that of an epicycloid referred to a vertex as origin; let us find a' and b', the radii of the fixed and rolling circles.

$$s = \frac{4b'(a'+b')}{a'}\sin\frac{a'}{a'+2b'}\tau, \qquad \text{by Art. 104 [2];}$$

hence,
$$\frac{4b'(a'+b')}{a'} = \frac{4b(a+b)}{a+2b},$$

$$\frac{a'}{a'+2b'} = \frac{a}{a+2b}.$$

Solving these equations, we get

$$a' = \frac{a^2}{a+2b},$$

$$b' = \frac{ab}{a+2b},$$

$$\frac{a'}{b'} = \frac{a}{b},$$

and the radii of the fixed and rolling circles have the same ratio in the evolute as in the original epicycloid; therefore the two curves are similar.

EXAMPLE.

Show that the evolute of a hypocycloid is a similar hypocycloid.

110. We have seen that in *involute* and *evolute* τ has the same value; that is, $\tau = \tau'$.

If s' and τ' refer to the *evolute*, and s and τ to the *involute*, we have found that

$$s' = \frac{ds}{d\tau}\bigg|_0^{\tau'},$$

or
$$s' = \frac{ds}{d\tau'} - l, \quad l \text{ being a constant,}$$

the length of the radius of curvature at the origin.

$$(s'+l)\,d\tau' = ds,$$

$$s = \int_0^\tau (s'+l)\,d\tau'$$

is the equation of the *involute*.

The involute of the catenary $s = a\tan\tau$ is, when $l = 0$,

$$s = a\int_0^\tau \tan\tau . d\tau = a\log\sec\tau, \quad \textit{the tractrix.}$$

The involute of the cycloid $s = 4a\sin\tau$ when $l = 0$ is

$$s = 4a\int_0^\tau \sin\tau . d\tau = 4a(1-\cos\tau),$$

an equal cycloid referred to its cusp as origin.

The involute of a cycloid referred to its cusp $s = 4a(1-\cos\tau)$ when $l = 0$ is

$$s = 4a\int_0^\tau (1-\cos\tau)\,d\tau = 4a(\tau + \sin\tau),$$

a curve we have not studied.

The involute of a circle $s = a\tau$ when $l = 0$ is

$$s = a\int_0^\tau \tau . d\tau = \frac{a\tau^2}{2}.$$

111. While any given curve has but one evolute, it has an infinite number of involutes, since the equation of the involute

$$s' = \int_0^\tau (s+l)\,d\tau$$

contains an arbitrary constant l; and the nature of the involute will in general be different for different values of l.

If we form the involute of a given curve, taking a particular value for l, and form the involute of this involute, taking the same value of l, and so on indefinitely, the curves obtained will continually approach the logarithmic spiral.

Let $$s = f\tau \tag{1}$$

be the given curve.

$$s = \int_0^\tau (l + f\tau)\,d\tau = l\tau + \int_0^\tau f\tau . d\tau$$

is the first involute;

$$s = \int_0^\tau \left(l + l\tau + \int_0^\tau f\tau . d\tau\right) d\tau = l\tau + \frac{l\tau^2}{2} + \int_0^\tau\int_0^\tau f\tau . d\tau^2$$

is the second involute;

$$s = l\tau + \frac{l\tau^2}{2} + \frac{l\tau^3}{3\,!} + \cdots\cdots + \frac{l\tau^n}{n\,!} + \int_0^{\tau_n} f\tau . d\tau^n \tag{2}$$

is the nth involute.

By Maclaurin's Theorem,

$$f\tau = fo + \tau f'o + \frac{\tau^2}{2\,!}f''o + \frac{\tau^3}{3\,!}f'''o + \cdots\cdots.$$

But $s = 0$ when $\tau = 0$; hence $fo = 0$, and

$$f\tau = A_1\tau \ + \frac{A_2}{2\,!}\tau^2 + \frac{A_3}{3\,!}\tau^3 + \cdots\cdots,$$

$$\int_0^\tau f\tau.d\tau \ = \frac{A_1}{2}\tau^2 + \frac{A_2}{3\,!}\tau^3 + \frac{A_3}{4\,!}\tau^4 + \cdots\cdots,$$

$$\int_0^{\tau_2} f\tau.d\tau^2 = \frac{A_1}{3\,!}\tau^3 + \frac{A_2}{4\,!}\tau^4 + \frac{A_3}{5\,!}\tau^5 + \cdots\cdots,$$

$$\int_0^{\tau_n} f\tau.d\tau^n = \frac{A_1\tau^{n+1}}{(n+1)\,!} + \frac{A_2\tau^{n+2}}{(n+2)\,!} + \frac{A_3\tau^{n+3}}{(n+3)\,!} + \cdots\cdots; \quad (3)$$

as n increases indefinitely all the terms of (3) approach zero (I. Art. 133), and the limiting form of (2) is

$$s = l\tau + \frac{l\tau^2}{2\,!} + \frac{l\tau^3}{3\,!} + \cdots\cdots$$

$$= l\left(1 + \frac{\tau}{1} + \frac{\tau^2}{2\,!} + \frac{\tau^3}{3\,!} + \cdots\cdots - 1\right),$$

$$s = l(e^\tau - 1) \qquad \text{by I. Art. 133 [2]},$$

which is a logarithmic spiral.

112. The equation of a curve in rectangular coördinates is readily obtained from the intrinsic equation.

Given $$s = f\tau,$$

we know that $$\sin\tau = \frac{dy}{ds},$$

and $$\cos\tau = \frac{dx}{ds};$$

hence $$dx = \cos\tau ds = \cos\tau f'\tau.d\tau,$$

$$dy = \sin\tau ds = \sin\tau f'\tau.d\tau,$$

$$\left.\begin{aligned} x &= \int_0^\tau \cos\tau f'\tau.d\tau \\ y &= \int_0^\tau \sin\tau f'\tau.d\tau \end{aligned}\right\}.$$

The elimination of τ between these equations will give us the equation of the curve in terms of x and y. Let us apply this method to the catenary.

$$s = a\tan\tau,$$

$$ds = a\sec^2\tau.d\tau,$$

$$x = a\int_0^\tau \sec\tau.d\tau = a\log\sqrt{\frac{1+\sin\tau}{1-\sin\tau}},$$

$$y = a\int_0^\tau \sec\tau\tan\tau.d\tau = a(\sec\tau - 1),$$

$$e^{\frac{2x}{a}} = \frac{1+\sin\tau}{1-\sin\tau},$$

$$\sin\tau = \frac{e^{\frac{2x}{a}}-1}{e^{\frac{2x}{a}}+1} = \frac{e^{\frac{x}{a}}-e^{-\frac{x}{a}}}{e^{\frac{x}{a}}+e^{-\frac{x}{a}}},$$

$$\sec\tau = \tfrac{1}{2}(e^{\frac{x}{a}}+e^{-\frac{x}{a}}),$$

$$y = \frac{a}{2}(e^{\frac{x}{a}}+e^{-\frac{x}{a}}) - a,$$

the equation of the catenary referred to its lowest point as origin.

Curves in Space.

113. The length of the arc of a curve of double curvature is the limit of the sum of the chords of smaller arcs into which the given arc may be broken up, as the number of these smaller arcs is indefinitely increased. Let (x, y, z), $(x + dx,\ y + \Delta y,\ z + \Delta z)$ be the coördinates of the extremities of any one of the small arcs in question; $dx, \Delta y, \Delta z$ are infinitesimal; $\sqrt{dx^2 + \Delta y^2 + \Delta z^2}$ is the length of the chord of the arc. In dealing with the limit of the sum of these chords, any one may be replaced by a quantity differing from it by infinitesimals of higher order than the first. $\sqrt{dx^2 + dy^2 + dz^2}$ is such a value;

hence $$s = \int_{x=x_0}^{x=x_1} \sqrt{dx^2 + dy^2 + dz^2}.$$

Let us rectify the helix.

$$\left.\begin{array}{l} x = a\cos\theta \\ y = a\sin\theta \\ z = k\theta \end{array}\right\}. \qquad \text{(I. Art. 214.)}$$

$$dx = -a\sin\theta . d\theta,$$

$$dy = a\cos\theta . d\theta,$$

$$dz = k d\theta,$$

$$ds^2 = (a^2 + k^2)\,d\theta^2,$$

$$s = (a^2 + k^2)^{\frac{1}{2}}\int_{\theta_0}^{\theta_1} d\theta = \sqrt{a^2 + k^2}\,(\theta_1 - \theta_0).$$

EXAMPLES.

(1) Find the length of the curve $\left(y = \dfrac{x^2}{2a},\ z = \dfrac{x^3}{6a^2}\right)$.

Ans. $s = x + z + l.$

(2) $y = 2\sqrt{ax} - x,\ z = x - \frac{2}{3}\sqrt{\dfrac{x^3}{a}}.$ *Ans.* $s = x + y - z.$

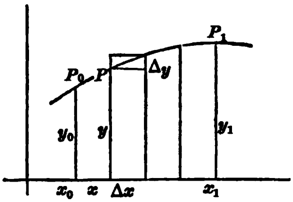
CHAPTER X.

AREAS.

114. We have found and used a formula for the area bounded by a given curve, the axis of X, and a pair of ordinates.

$$A = \int y dx.$$

We can readily get this formula as a definite integral. The area in the figure is the sum of the slices into which it is divided by the ordinates; if Δx, the base of each slice, is indefinitely decreased, the slice is infinitesimal. The area of any slice differs from $y\Delta x$ by less than $\Delta y \Delta x$, which is of the second order if Δx is the principal infinitesimal. We have then



$$A = \underset{\Delta x \doteq 0}{\text{limit}} \sum_{x=x_0}^{x=x_1} y\Delta x, \qquad \text{by I. Art. 161.}$$

Hence
$$A = \int_{x_0}^{x_1} y dx.$$

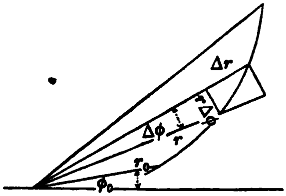
EXAMPLES.

(1) Show that $\int_{y_0}^{y_1} x dy$ is the area bounded by a curve, the axis of Y, and perpendiculars let fall from the ends of the bounding arc upon the axis of Y.

(2) If the axes are inclined at the angle ω, show that these formulas become

$$A = \sin\omega \int_{x_0}^{x_1} y dx = \sin\omega \int_{y_0}^{y_1} x dy.$$

115. In polar coördinates we can regard the area between two radii vectores and the curve as the limit of the sum of sectors.



The area in question is the sum of the smaller sectorial areas, any one of which differs from $\frac{1}{2}r^2\Delta\phi$ by less than the difference between the two circular sectors $\frac{1}{2}(r+\Delta r)^2\Delta\phi$ and $\frac{1}{2}r^2\Delta\phi$; that is, by less than $r\Delta r\Delta\phi+\frac{(\Delta r)^2\Delta\phi}{2}$, which is of the second order if $\Delta\phi$ is the principal infinitesimal.

Hence
$$A=\underset{\Delta\phi\doteq 0}{\text{limit}}\left[\frac{1}{2}\sum_{\phi=\phi_0}^{\phi=\phi_1}r^2\Delta\phi\right],$$
$$A=\frac{1}{2}\int_{\phi_0}^{\phi_1}r^2d\phi.$$

116. Let us find the area between the catenary, the axis of X, the axis of Y, and any ordinate.

$$A=\int^x ydx=\frac{a}{2}\int_0^x(e^{\frac{x}{a}}+e^{-\frac{x}{a}})dx,$$
$$A=\frac{a^2}{2}(e^{\frac{x}{a}}-e^{-\frac{x}{a}}),$$

but
$$\frac{a}{2}(e^{\frac{x}{a}}-e^{-\frac{x}{a}})=s, \qquad \text{by Art. 90.}$$

Hence
$$A=as,$$

and the area in question is the length of the arc multiplied by the distance of the lowest point of the curve from the origin.

117. Let us find the area between the tractrix and the axis of X.

We have
$$dx=-\frac{dy}{y}\sqrt{a^2-y^2}. \qquad \text{(Art. 91.)}$$
$$A=\int ydx=-\int dy\sqrt{a^2-y^2}.$$

The area in question is

$$A = -\int_a^0 dy \sqrt{a^2 - y^2} = \frac{\pi a^2}{4},$$

which is the area of the quadrant of a circle with a as radius.

EXAMPLE.

Give, by the aid of infinitesimals, a geometric proof of the result just obtained for the tractrix.

118. In the last section we found the area between a curve and its asymptote, and obtained a finite result. Of course this means that, as our second bounding ordinate recedes from the origin, the area in question, instead of increasing indefinitely, approaches a finite limit, which is the area obtained. Whether the area between a curve and its asymptote is finite or infinite will depend upon the nature of the curve.

Let us find the area between an hyperbola and its asymptote.

The equation of the hyperbola referred to its asymptotes as axes is

$$xy = \frac{a^2 + b^2}{4}.$$

Let ω be the angle between the asymptotes; then

$$A = \sin\omega \int_0^\infty y dx = \frac{a^2 + b^2}{4} \sin\omega \int_0^\infty \frac{dx}{x} = \infty .$$

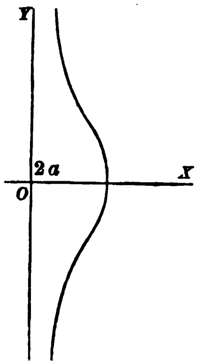
Take the curve $$y^2 x = 4a^2(2a - x),$$

or $$y^2 = 4a^2 . \frac{2a - x}{x};$$

any value of x will give two values of y equal with opposite signs; therefore the axis of x is an axis of symmetry of the curve.

When $x = 2a$, $y = 0$; as x decreases, y increases; and when $x = 0$, $y = \infty$. If x is negative, or greater than $2a$, y is imaginary. The shape of the curve is something like that in the

figure, the axis of Y being an asymptote. The area between the curve and the asymptote is then either



$$A = 2\int_0^{2a} y dx \quad \text{or} \quad A = 2\int_0^{\infty} x dy;$$

by the first formula,

$$A = 4a\int_0^{2a} \sqrt{\frac{2a - x}{x}} \cdot dx = 4a^2\pi;$$

by the second,

$$A = 16a^3 \int_0^{\infty} \frac{dy}{y^2 + 4a^2} = 4a^2\pi.$$

EXAMPLES.

(1) Find the area between the curve $y^2(x^2 + a^2) = a^2x^2$ and its asymptote $y = a$. *Ans.* $A = 2a^2$.

(2) Find the area between $y^2(2a - x) = x^3$ and its asymptote $x = 2a$. *Ans.* $A = 3\pi a^2$.

(3) Find the area bounded by the curve $y^2 = \frac{x^2(a + x)}{a - x}$ and its asymptote $x = a$.

Ans. $A = 2a^2\left(1 + \frac{\pi}{4}\right)$.

119. If the coördinates of the points of a curve are expressed in terms of an auxiliary variable, no new difficulty is presented.

Take the case of the circle $x^2 + y^2 = a^2$, which may be written

$$\left.\begin{array}{l} x = a\cos\phi \\ y = a\sin\phi \end{array}\right\};$$

$$dy = a\cos\phi d\phi.$$

The whole area $A = a^2\int_0^{2\pi} \cos^2\phi d\phi = \pi a^2$.

EXAMPLES.

(1) The whole area of an ellipse $\left.\begin{array}{l} x = a\cos\phi \\ y = b\sin\phi \end{array}\right\}$ is πab.

(2) The area of an arch of the cycloid is $3\pi a^2$.

(3) The area of an arch of the companion to the cycloid $x = a\theta$, $y = a(1 - \cos\theta)$ is $2\pi a^2$.

120. If we wish to find the area between two curves, or the area bounded by a closed curve, the altitude of our elementary rectangle is the difference between the two values of y, which correspond to a single value of x. If the area between two curves is required, we must find the abscissas of their points of intersection, and they will be our limits of integration; if the whole area bounded by a closed curve is required, we must find the values of x belonging to the points of contact of tangents parallel to the axis of Y.

Let us find the whole area of the curve

$$a^4y^2 + b^2x^4 = a^2b^2x^2,$$

or

$$a^4y^2 = b^2x^2(a^2 - x^2).$$

The curve is symmetrical with reference to the axis of X, and passes through the origin. It consists of two loops whose areas must be found separately. Let us find where the tangents are parallel to the axis of Y.

$$y = \frac{b}{a^2}x\sqrt{a^2 - x^2},$$

$$\frac{dy}{dx} = \frac{b}{a^2}\cdot\frac{a^2 - 2x^2}{\sqrt{a^2 - x^2}} = \tan\tau.$$

$\tau = \frac{\pi}{2}$ when $\tan\tau = \infty$, that is, when $x = \pm a$.

$$A = 2\frac{b}{a^2}\int_{-a}^{0} x\sqrt{a^2 - x^2}.dx + 2\frac{b}{a^2}\int_0^a x\sqrt{a^2 - x^2}.dx = \tfrac{4}{3}ab.$$

Again; find the whole area of $(y-x)^2=a^2-x^2$.

$$y=x\pm\sqrt{a^2-x^2},$$

$$A=\int(y'-y'')\,dx=\int 2\sqrt{a^2-x^2}.$$

To find the limits of integration, we must see where $\tau=\frac{\pi}{2}$.

$$\frac{dy}{dx}=\frac{\sqrt{a^2-x^2}\mp x}{\sqrt{a^2-x^2}}=\infty \text{ when } x=\pm a.$$

$$A=2\int_{-a}^{a}\sqrt{a^2-x^2}=\pi a^2.$$

EXAMPLES.

(1) Find the area of the loop of the curve $y^2=\frac{x^2(a+x)}{a-x}$.

Ans. $2a^2\left(1-\frac{\pi}{4}\right)$.

(2) Find the area between the curves $y^2-4ax=0$ and $x^3-4ay=0$.

Ans. $\frac{16a^2}{3}$.

(3) Find the aréa of a loop of $a^2y^4=x^4(a^2-x^2)$. *Ans.* $\frac{4a^2}{5}$.

(4) Find the whole area of the curve

$$2y^2(a^2+x^2)-4ay(a^2-x^2)+(a^2-x^2)^2=0.$$

Ans. $a^2\pi\left(4-\frac{5\sqrt{2}}{2}\right)$.

121. We have seen that in polar coördinates

$$A=\tfrac{1}{2}\int_{\varphi_0}^{\varphi_1}r^2\,d\phi.$$

Let us try one or two examples.

(a) To find the whole area of a circle.

The polar equation is $r=a$.

$$A=\tfrac{1}{2}\int_0^{2\pi}a^2\,d\phi=\pi a^2.$$

(*b*) To find the area of the cardioide $r = 2a(1-\cos\phi)$.

$$A = \tfrac{1}{2}\int_0^{2\pi} 4a^2(1-\cos\phi)^2 d\phi = 2a^2\int_0^{2\pi}(1-2\cos\phi+\cos^2 d\phi)d\phi,$$

$$A = 6a^2\pi.$$

(*c*) To find the area between an arch of the epicycloid and the circumference of the fixed circle.

$$\left.\begin{aligned} x &= (a+b)\cos\theta - b\cos\frac{a+b}{b}\theta \\ y &= (a+b)\sin\theta - b\sin\frac{a+b}{b}\theta \end{aligned}\right\}.$$

We can get the area bounded by two radii vectores and the arch in question, and subtract the area of the corresponding sector of the fixed circle.

Changing to polar coördinates,

$$x = r\cos\phi,$$
$$y = r\sin\phi.$$

We want $\frac{1}{2}\int r^2 d\phi$.

$$\tan\phi = \frac{y}{x},$$

$$\sec^2\phi d\phi = \frac{xdy - ydx}{x^2};$$

but, since $\quad x = r\cos\phi, \quad \sec\phi = \frac{r}{x};$

hence $\quad \frac{r^2 d\phi}{x^2} = \frac{xdy - ydx}{x^2},$

and $\quad r^2 d\phi = xdy - ydx;$

$$dx = (a+b)\left(-\sin\theta + \sin\frac{a+b}{b}\theta\right)d\theta,$$

$$dy = (a+b)\left(\cos\theta - \cos\frac{a+b}{b}\theta\right)d\theta.$$

$$xdy - ydx = (a+b)(a+2b)\left(1 - \cos\frac{a}{b}\theta\right)d\theta = r^2 d\phi.$$

Our limits of integration are obviously 0 and $\frac{2b\pi}{a}$.

Hence $$A = \tfrac{1}{2}(a+b)(a+2b)\int_0^{\frac{2b\pi}{a}}\left(1-\cos\frac{a}{b}\theta\right)d\theta,$$

$$A = \frac{b\pi}{a}(a+b)(a+2b),$$

is the area of the sector of the epicycloid. Subtract the area of the circular sector πab, and we get

$$A = \frac{b^2(3a+2b)}{a}\pi$$

as the area in question.

(d) To find the area of a loop of the curve $r^2 = a^2\cos 2\phi$.

For any value of ϕ the values of r are equal with opposite signs. Hence the origin is a centre.

When $\phi = 0$, $r = \pm a$; as θ increases, r decreases in length till $\phi = \frac{\pi}{4}$, when $r = 0$; as soon as $\phi > \frac{\pi}{4}$, r is imaginary. If ϕ decreases from 0, r decreases in length until $\phi = -\frac{\pi}{4}$, when $r = 0$; and when $\phi < \frac{\pi}{4}$, r is imaginary. To get the area of a loop, then, we must integrate from $\phi = -\frac{\pi}{4}$ to $\phi = \frac{\pi}{4}$.

$$A = \tfrac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r^2 d\phi = \tfrac{1}{2}a^2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\cos 2\phi . d\phi = \frac{a^2}{2}.$$

EXAMPLES.

(1) Find the area of a sector of the parabola $r = \dfrac{m}{1+\cos\phi}$.

(2) Find the area of a loop of the curve $r^2\cos\phi = a^2\sin 3\phi$.

Ans. $\dfrac{3a^2}{4} - \dfrac{a^2}{2}\log 2$.

(3) Find the whole area of the curve $r = a(\cos 2\phi + \sin 2\phi)$.

Ans. πa^2.

(4) Find the area of a loop of the curve $r\cos\phi = a\cos 2\phi$.

Ans. $\left(2 - \dfrac{\pi}{2}\right)a^2$.

(5) Find the area between $r = a(\sec\phi + \tan\phi)$ and its asymptote $r\cos\phi = 2a$.

Ans. $\left(\dfrac{\pi}{2} + 2\right)a^2$.

122. When the equation of a curve is given in rectangular coördinates, we can often simplify the problem of finding its area by transforming to polar coördinates.

For example, let us find the area of

$$(x^2+y^2)^2=4a^2x^2+4b^2y^2.$$

Transform to polar coördinates.

$$r^4=4r^2(a^2\cos^2\phi+b^2\sin^2\phi),$$

$$r^2=4\,(a^2\cos^2\phi+b^2\sin^2\phi),$$

$$A=2\int_0^{2\pi}(a^2\cos^2\phi+b^2\sin^2\phi)\,d\phi=2\pi(a^2+b^2).$$

EXAMPLES.

(1) Find the area of a loop of the curve $(x^2+y^2)^3=4a^2x^2y^2$.

Ans. $\dfrac{\pi a^2}{8}$.

(2) Find the whole area of the curve $\dfrac{x^2}{a^4}+\dfrac{y^2}{b^4}=\dfrac{1}{c^2}\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\right)^2$.

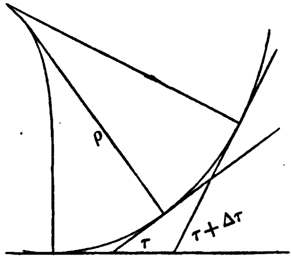
Ans. $\dfrac{\pi c^2}{2ab}(a^2+b^2)$.

(3) Find the area of a loop of the curve $y^3-3axy+x^3=0$.

Ans. $\dfrac{3a^2}{2}$.

123. The area between a curve and its evolute can easily be found from the intrinsic equation of the curve.

It is easily seen that the area bounded by the radii of curvature at two points infinitely near, by the curve and by the evolute, differs from $\frac{1}{2}\rho^2 d\tau$ by an infinitesimal of higher order. The area bounded by two given radii vectores, the curve and the evolute, is then



$$A=\tfrac{1}{2}\int_{\tau_0}^{\tau_1}\rho^2 d\tau.$$

$$\rho = \frac{ds}{d\tau}.$$

Hence $$A = \tfrac{1}{2}\int_{\tau_0}^{\tau_1}\left(\frac{ds}{d\tau}\right)^2 d\tau.$$

For example, the area between a cycloid and its evolute is

$$A = \tfrac{1}{2}\int_{\tau_0}^{\tau_1}\left(\frac{d(4a\sin\tau)}{d\tau}\right)^2 d\tau$$

$$= 8a^2\int_{\tau_0}^{\tau_1}\cos^2\tau d\tau.$$

Let $$\tau_0 = 0 \quad \text{and} \quad \tau_1 = \frac{\pi}{2};$$

$$A = 8a^2\int_0^{\frac{\pi}{2}}\cos^2\tau d\tau = 2\pi a^2.$$

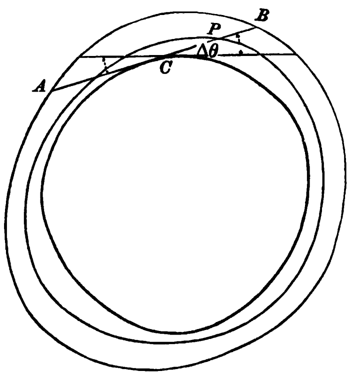
EXAMPLES.

(1) Find the area between a circle and its evolute.

(2) Find the area between the circle and its involute.

Holditch's Theorem.

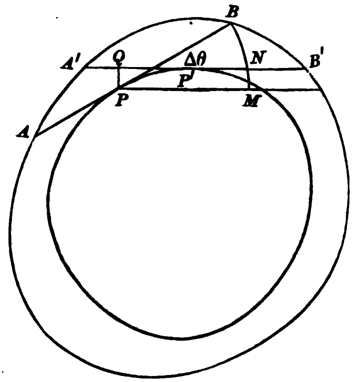
124. If a line of fixed length move with its ends on any curve which is always concave toward it, the area between the curve and the locus of a given point of the moving line is equal to the area of an ellipse, of which the segments into which the line is divided by the given point are the semi-axes.



Let the figure represent the given curve, the locus of P, and the envelope of the moving line.

Let $AP = a$ and $PB = b$, and let $CB = \rho$, C being the point of contact of the moving line with its envelope. Let $AB = a + b = c$.

The area between the first curve and the second is the area between the first curve and the envelope, minus the area between the second curve and the envelope.



Let θ be the angle which the moving line makes at any instant with some fixed direction. Let the figure represent two near positions of the moving line; $\Delta\theta$, the angle between these positions, being the principal infinitesimal.

$PB=\rho,\ P'B'=\rho+\Delta\rho.$

The area $PBB'P'P$ differs from $\frac{1}{2}\rho^2 d\theta$ by an infinitesimal of higher order than the first.

$\frac{1}{2}\rho^2 d\theta$ is the area of $PBMP$, and differs from $PP'NB$ by less than the rectangle on PM and PQ, which is of higher order than the first, by I. Art. 153. But $PP'NB$ differs from $PP'B'B$ by less than the rectangle on BN and NB', which is of higher order than the first, since NB', which is less than $PP'+\Delta\rho$, is infinitesimal and $\Delta\theta$ is infinitesimal.

The area between the first curve and the envelope is then $\frac{1}{2}\int_0^{2\pi}\rho^2 d\theta$; or, since we can take $PP'A'A$ just as well for our elementary area, $\frac{1}{2}\int_0^{2\pi}(c-\rho)^2 d\theta$.

Hence $$\frac{1}{2}\int_0^{2\pi}\rho^2 d\theta=\frac{1}{2}\int_0^{2\pi}(c-\rho)^2 d\theta;$$

whence $$2c\int_0^{2\pi}\rho d\theta=2c^2\pi,$$

or $$\int_0^{2\pi}\rho d\theta=\pi c. \tag{1}$$

The area between the second curve and the envelope is $\frac{1}{2}\int_0^{2\pi}(\rho-b)^2 d\theta$.

The area between the first curve and the second is then

$$A = \tfrac{1}{2}\int_0^{2\pi}\rho^2 d\theta - \tfrac{1}{2}\int_0^{2\pi}(\rho - b)^2 d\theta$$

$$= b\int_0^{2\pi}\rho d\theta - b^2\pi$$

$$= \pi bc - b^2\pi \qquad \text{by (1)},$$

$$= \pi b(a + b) - b^2\pi,$$

$$A = \pi ab, \qquad (2)$$

which is the area of an ellipse of which a and b are semi-axes.

Q. E. D.

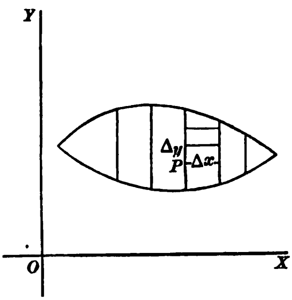
EXAMPLES.

(1) If a line of fixed length move with its extremities on two lines at right angles with each other, the area of the locus of a given point of the line is that of an ellipse on the segments of the line as semi-axes.

(2) The result of (1) holds even when the fixed lines are not perpendicular.

Areas by Double Integration.

125. If we choose to regard x and y as independent variables, we can find the area bounded by two given curves, $y = fx$ and $y = Fx$, by a double integration. Suppose the area in question divided into slices by lines drawn parallel to the axis of Y, and these slices subdivided into parallelograms by lines drawn parallel to the axis of X. The area of any one of the small parallelograms is $\Delta y \Delta x$. If we keep x constant, and take the sum of these rectangles from $y = fx$ to $y = Fx$, we shall get a result differing from the area of the corresponding slice by less than



$2\,\Delta x \Delta y$, which is infinitesimal of the second order if Δx and Δy are of the first order.

Hence $$\int_{fx}^{Fx} \Delta x . dy = \Delta x \int_{fx}^{Fx} dy$$

is the area of the slice in question. If now we take the limit of the sum of all these slices, choosing our initial and final values of x, so that we shall include the whole area, we shall get the area required.

Hence $$A = \int_{x_0}^{x_1} \left(\int_{fx}^{Fx} dy \right) dx.$$

In writing a double integral, the parentheses are usually omitted for the sake of conciseness, and this formula is given as

$$A = \int_{x_0}^{x_1} \int_{fx}^{Fx} dy dx,$$

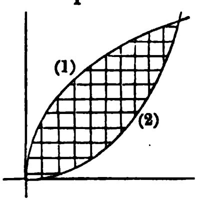
the order in which the integrations are to be performed being the same as if the parentheses were actually written.

If we begin by keeping y constant, and integrating with respect to x, we shall get the area of a slice formed by lines parallel to the axis of X, and we shall have to take the limit of the sum of these slices varying y in such a way as to include the whole area desired. In that case we should use the formula

$$A = \int_{y_0}^{y_1} \int_{f^{-1}y}^{F^{-1}y} dx dy.$$

126. For example, let us find the area bounded by the parabolas $y^2 = 4\,ax$ and $x^2 = 4\,ay$.

The parabolas intersect at the origin and at the point $(4\,a,\ 4\,a)$.



$$A = \int_0^{4a} \int_{\frac{x^2}{4a}}^{\sqrt{4ax}} dy dx, \quad \text{or} \quad A = \int_0^{4a} \int_{\frac{y^2}{4a}}^{\sqrt{4ay}} dx dy;$$

$$\int_{\frac{x^2}{4a}}^{\sqrt{4ax}} dy = \sqrt{4\,ax} - \frac{x^2}{4\,a};$$

$$\int_0^{4a} \int_{\frac{x^2}{4a}}^{\sqrt{4ax}} dy dx = \int_0^{4a} \left(\sqrt{4\,ax} - \frac{x^2}{4\,a} \right) dx = \frac{16}{3} a^2.$$

The second formula gives the same result.

EXAMPLES.

(1) Find the area of a rectangle by double integration; of a parallelogram; of a triangle.

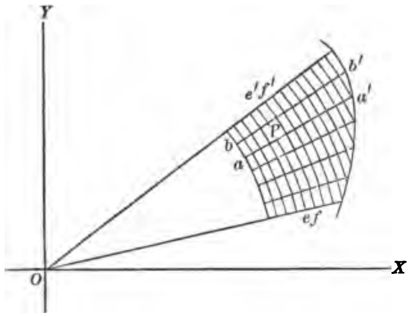
(2) Find the area between the parabola $y^2 = ax$ and the circle $y^2 = 2ax - x^2$.

Ans. $2\left(\frac{\pi a^2}{4} - \frac{2a^2}{3}\right)$.

(3) Find the whole area of the curve $(y - mx - c)^2 = a^2 - x^2$.

Ans. πa^2.

127. If we use polar coördinates we can still find our areas by double integration.



Let $r = f\phi$ and $r = F\phi$ be two curves. Divide the area between them into slices by drawing radii vectores; then subdivide these slices by drawing arcs of circles, with the origin as centre.

Let P, with coördinates r and ϕ, be any point within the space whose area is sought. The curvilinear rectangle at P has the base $r\Delta\phi$ and the altitude Δr; its area differs from $r\Delta\phi\Delta r$ by an infinitesimal of higher order than $r\Delta\phi\Delta r$.

The area of any slice as $aba'b'$ is $\int_{f\phi}^{F\phi} r\Delta\phi dr$, ϕ and $\Delta\phi$ being constant, that is $\Delta\phi\int_{f\phi}^{F\phi} rdr$. The whole area, the limit of the sum of such slices is $A = \int_{\phi_0}^{\phi_1}\int_{f\phi}^{F\phi} rdrd\phi$. (1)

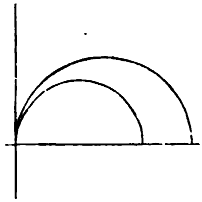
Or we may first sum our rectangles, keeping r unchanged, and we get as the area of $efe'f'$

$$r\Delta r\int_{f^{-1}\phi}^{F^{-1}\phi} d\phi, \text{ and } A = \int_{r_0}^{r_1}\int_{f^{-1}\phi}^{F^{-1}\phi} rd\phi dr. \qquad (2)$$

For example, the area between two concentric circles, $r = a$ and $r = b$, is

$$A = \int_b^a \int_0^{2\pi} r d\phi dr = \int_0^{2\pi} \int_b^a r dr d\phi = \pi(a^2 - b^2).$$

Again, let us find the area between two tangent circles and a diameter through the point of contact.



Let a and b be the two radii,

$$r = 2a\cos\phi \qquad (1)$$

and

$$r = 2b\cos\phi \qquad (2)$$

are the equations of the two circles.

$$A = \int_0^{\frac{\pi}{2}} \int_{2b\cos\phi}^{2a\cos\phi} r dr d\phi = 2(a^2 - b^2)\int_0^{\frac{\pi}{2}} \cos^2\phi d\phi = \frac{\pi}{2}(a^2 - b^2).$$

If we wish to reverse the order of our integrations we must break our area into two parts by an arc described from the origin as a centre, and with $2b$ as a radius; then we have

$$A = \int_0^{2b} \int_{\cos^{-1}\frac{r}{2b}}^{\cos^{-1}\frac{r}{2a}} r d\phi dr + \int_{2b}^{2a} \int_0^{\cos^{-1}\frac{r}{2a}} r d\phi dr$$

$$= \int_0^{2b} r\left(\cos^{-1}\frac{r}{2a} - \cos^{-1}\frac{r}{2b}\right) dr + \int_{2b}^{2a} r\cos^{-1}\frac{r}{2a} dr$$

$$= \frac{\pi}{2}(a^2 - b^2).$$

EXAMPLE.

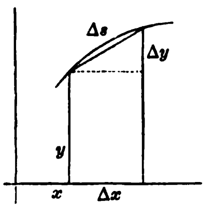
Find the area between the axis of X and two coils of the spiral $r = a\phi$.

CHAPTER XI.

AREAS OF SURFACES.

Surfaces of Revolution.

128. If a plane curve $y = fx$ revolves about the axis of X, the area of the surface generated is the limit of the sum of the areas generated by the chords of the infinitesimal arcs into which the whole arc may be broken up. Each of these chords will generate the surface of the frustum of a cone of revolution if it revolves completely around the axis; and the area of the surface of a frustum of a cone of revolution is, by elementary Geometry, one-half the sum of the circumferences of the bases multiplied by the slant height. The frustum generated by the chord in the figure will have an area differing by infinitesimals of higher order from $\pi(y + y + \Delta y)\Delta s$ or from $2\pi y ds$. The area generated by any given arc is then



$$S = 2\pi \int_{y_0}^{y_1} y ds. \qquad [1]$$

If the arc revolves through an angle θ instead of making a complete revolution, the surface generated is

$$S = \theta \int_{y_0}^{y} y ds. \qquad [2]$$

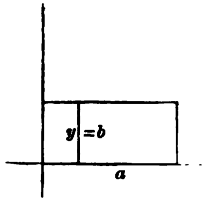
EXAMPLE.

Show that if the arc revolves about the axis of Y,

$$S = 2\pi \int_{x_0}^{x_1} x ds.$$

129. To find the area of a cylinder of revolution.

Take the axis of the cylinder as the axis of X. Let a be the altitude and b the radius of the base of the cylinder. The equation of the revolving line is



$$y = b;$$

$$dy = 0,$$

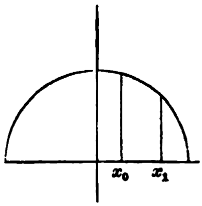
$$ds = \sqrt{dx^2 + dy^2} = dx;$$

$$S = 2\pi\int_0^a y dx = 2\pi ab,$$

or the product of the altitude by the circumference of the base.

Again, let us find the surface of a zone.

The equation of the generating circle is



$$x^2 + y^2 = a^2;$$

$$ds = \frac{a dx}{y};$$

$$S = 2\pi\int_{x_0}^{x_1} a dx = 2a\pi(x_1 - x_0).$$

If $x_0 = -a$ and $x_1 = a$,

$$S = 4a^2\pi.$$

Hence the surface of a zone is the altitude of the zone multiplied by the circumference of a great circle, and the surface of a sphere is equal to the areas of four great circles.

Again, take the surface generated by the revolution of a cycloid about its base.

$$\left.\begin{aligned} x &= a\theta - a\sin\theta \\ y &= a - a\cos\theta \end{aligned}\right\};$$

$$ds = a d\theta\sqrt{2(1 - \cos\theta)}, \qquad \text{by Art. 94};$$

$$S = 2\pi\int_0^{2\pi} a^2\sqrt{2}\,.(1 - \cos\theta)^{\frac{3}{2}} d\theta = \tfrac{64}{3}\pi a^2.$$

EXAMPLES.

(1) The area of the surface generated by the revolution of the ellipse
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$
about the axis of X is $2\pi ab\left(\sqrt{1-e^2}+\frac{\sin^{-1}e}{e}\right)$;

about the axis of Y is $2\pi a^2\left(1+\frac{1-e^2}{2e}\log\frac{1+e}{1-e}\right)$,

where $e^2=\frac{a^2-b^2}{a^2}$.

(2) Find the area of the surface generated by the revolution of the catenary about the axis of X; about the axis of Y.

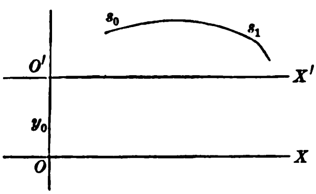
(3) The whole surface generated by the revolution of the tractrix about its asymptote is $4\pi a^2$.

(4) The area generated by the revolution of a cycloid about its vertical axis is $8\pi a^2(\pi-\frac{4}{3})$.

(5) The area generated by the revolution of a cycloid about the tangent at its vertex is $\frac{32}{3}\pi a^2$.

130. If we know the area generated by the revolution of a curve about any axis, we can get the area generated by the revolution about any parallel axis by an easy transformation of coördinates.

Given the surface generated by the arc from s_0 to s_1 about OX, to find the area generated by the same arc when it revolves about $O'X'$.



Let S be the surface about OX, and S' about $O'X'$.

We have
$$S=2\pi\int_{s_0}^{s_1}y\,ds,\quad S'=2\pi\int_{s_0}^{s_1}y'ds'.$$

By Anal. Geom., $x = x'$,

$$y = y_0 + y'.$$

Hence $dx = dx', \quad dy = dy', \quad ds = ds',$

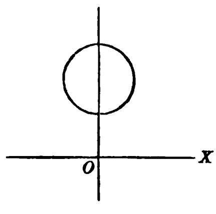
and $$S = 2\pi \int_{s_0}^{s_1} (y_0 + y')ds = 2\pi y_0(s_1 - s_0) + 2\pi \int_{s_0}^{s_1} y'ds,$$

$$= 2\pi y_0(s_1 - s_0) + S'.$$

Therefore $$S' = S - 2\pi y_0(s_1 - s_0). \qquad [1]$$

$s_1 - s_0$ is the length of the revolving curve; $2\pi y_0$ is the circumference of a circle of which y_0 is the radius. Hence the new area is equal to the old area minus the area of a cylinder whose length is the length of the given arc and whose base is a circle of which the distance between the two lines is radius.

In using this principle careful attention must be paid to the sign of y_0, and it must be noted that the original formula $S = 2\pi \int_{s_0}^{s_1} yds$ will always give a negative value for the area of the surface generated, if the revolving arc starts from below the axis; and hence, that the surface generated by the revolution of any curve about an axis of symmetry will come out zero.



As an example of the use of the principle, let us find the surface of a ring.

Let a be the distance of the centre of the circle from the axis, and b the radius of the circle. Since the area generated by the revolution of the circle about a diameter is zero, the required area is

$$2\pi b . 2\pi a = 4\pi^2 ab.$$

EXAMPLE.

Find the area of the ring generated by the revolution of a cycloid about any axis parallel to its base.

Ans. $S = 4ab\pi\left(\pi + \dfrac{16a + 12b}{3b}\right).$

131. If we use polar coördinates,

$$S = 2\pi \int_{s_0}^{s_1} y ds$$

becomes
$$S = 2\pi \int_{s_0}^{s_1} r \sin\phi . ds.$$

where
$$ds = \sqrt{dr^2 + r^2 d\phi^2}.$$

For example; let us find the area of the surface generated by the revolution of the upper half of a cardioide about the horizontal axis.

$$r = 2a(1 - \cos\phi);$$

$$dr = 2a \sin\phi . d\phi,$$

$$ds^2 = 8a^2(1 - \cos\phi) d\phi^2,$$

$$S = 2\pi \int_0^{\pi} 4\sqrt{2}\, a^2 (1 - \cos\phi)^{\frac{3}{2}} \sin\phi . d\phi.$$

$$S = \tfrac{128}{5} \pi a^2.$$

EXAMPLES.

(1) Find the surface of a sphere from the polar equation.

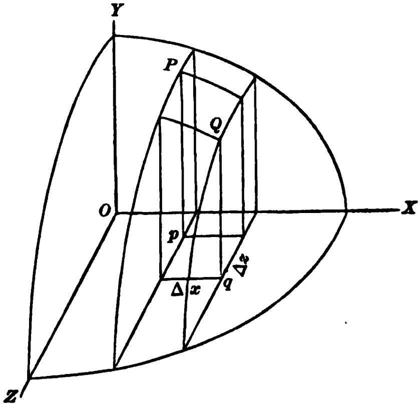
(2) Find the surface of a paraboloid of revolution from the polar equation of the parabola

$$r = \frac{m}{1 - \cos\phi}.$$

Any Surface.

132. Let x, y, z be the coördinates of any point P of the surface, and $x + \Delta x$, $y + \Delta y$, $z + \Delta z$ the coördinates of a second point Q infinitely near the first. Draw planes through P and Q parallel to the planes of XY and YZ. These planes will intercept a curved quadrilateral PQ on the surface; its projection pq, a rectangle, on the plane of XZ; and a parallelogram $p'q'$ not shown in the figure on the tangent plane at P, of which pq is

the projection. PQ will differ from $p'q'$ by an infinitesimal of higher order, and therefore our required surface will be the limit of the sum of the parallelograms of which $p'q'$ is any one.



If β is the angle the tangent plane at P makes with XZ, $p'q'\cos\beta = pq$ or $p'q' = pq\sec\beta. = \Delta x\Delta z\sec\beta$, and σ, our surface required, is equal to the double integral $\sigma = \int\int \sec\beta dx dz$ taken between limits so chosen as to embrace the whole surface.

The equation of the tangent plane is

$(x-x_0)D_{x_0}f + (y-y_0)D_{y_0}f + (z-z_0)D_{z_0}f = 0$, by I. Art. 217,

(x_0,y_0,z_0) standing for the coördinates of the point of contact, and $f(x,y,z)=0$ being the equation of the surface.

The direction cosines of the perpendicular from the origin upon the plane are

$$\cos a = \frac{D_{x_0}f}{\sqrt{(D_{x_0}f)^2 + (D_{y_0}f)^2 + (D_{z_0}f)^2}},$$

$$\cos\beta = \frac{D_{y_0}f}{\sqrt{(D_{x_0}f)^2 + (D_{y_0}f)^2 + (D_{z_0}f)^2}},$$

$$\cos\gamma = \frac{D_{z_0}f}{\sqrt{(D_{x_0}f)^2 + (D_{y_0}f)^2 + (D_{z_0}f)^2}},$$

by Anal. Geom. of Three Dimensions.

Hence, dropping the accents,

$$\sigma = \int\int \frac{\sqrt{(D_x f)^2 + (D_y f)^2 + (D_z f)^2}}{D_y f} dx dz. \qquad [1]$$

By considering the projections upon the other coördinate planes we shall find

$$\sigma = \int\int \frac{\sqrt{(D_x f)^2 + (D_y f)^2 + (D_z f)^2}}{D_x f} dy dz; \qquad [2]$$

$$\sigma = \int\int \frac{\sqrt{(D_x f)^2 + (D_y f)^2 + (D_z f)^2}}{D_z f} dx dy. \qquad [3]$$

In each of the formulas the derivatives are partial derivatives.

Let us find the area of the portion of the surface of the sphere

$$x^2 + y^2 + z^2 = a^2$$

intercepted by the three coördinate planes.

$$D_x f = 2x,$$

$$D_y f = 2y,$$

$$D_z f = 2z,$$

$$\sqrt{(D_x f)^2 + (D_y f)^2 + (D_z f)^2} = 2a.$$

$$\sigma = \int_0^a \int_0^{\sqrt{a^2 - z^2}} \frac{a}{x} dy dz; \qquad (1)$$

or

$$\sigma = \int_0^a \int_0^{\sqrt{a^2 - x^2}} \frac{a}{y} dz dx; \qquad (2)$$

or

$$\sigma = \int_0^a \int_0^{\sqrt{a^2 - y^2}} \frac{a}{z} dx dy. \qquad (3)$$

For, in the second one, which agrees best with the figure, we must take our limits so that the limit of the sum of the projections may be the quadrant in which the sphere is cut by the

plane XZ; and the equation of this section is obtained by letting $y = 0$ in the equation of the sphere, and is

$$x^2 + z^2 = a^2,$$

whence $$z = \sqrt{a^2 - x^2}.$$

If we take as our limits in the integral $\int \frac{a}{y} dz$ zero and $\sqrt{a^2 - x^2}$ we shall get the area whose projection is a strip running from the axis of Z to the curve; then, taking $\int \left(\int \frac{a}{y} dz \right) dx$ from 0 to a, we shall get the area whose projection is the sum of all these strips, and that is our required surface.

$$y = \sqrt{a^2 - x^2 - z^2},$$

$$\sigma = a \int_0^a \int_0^{\sqrt{a^2 - x^2}} \frac{dz dx}{\sqrt{a^2 - x^2 - z^2}};$$

$$\int \frac{dz}{\sqrt{a^2 - x^2 - z^2}} = \sin^{-1} \frac{z}{\sqrt{a^2 - x^2}}$$

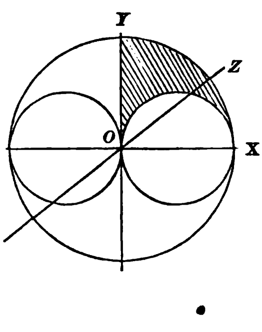
if we regard x as constant;

$$\int_0^{\sqrt{a^2 - x^2}} \frac{dz}{\sqrt{a^2 - x^2 - z^2}} = \frac{\pi}{2};$$

$$\sigma = a \int_0^a \frac{\pi}{2} dx = \frac{\pi a^2}{2},$$

the required area. Formulas (1) and (3) give the same result.

133. Suppose two cylinders of revolution drawn tangent to each other, and perpendicular to the plane of a great circle of a sphere, each having the radius of the great circle as a diameter; required the surface of the sphere not included by the cylinders.



The surface required is eight times the surface of which the shaded portion of the figure is the projection.

If we take the plane of the great circle as the plane of XY,

$$x^2 - ax + y^2 = 0 \qquad (1)$$

is the equation of the cylinder, and

$$x^2 + y^2 + z^2 = a^2 \qquad (2)$$

of the sphere.

We have $\sigma = \int\int \frac{\sqrt{(D_x f)^2 + (D_y f)^2 + (D_z f)^2}}{D_z f} dydx.$

From (2)

$$D_x f = 2x,$$

$$D_y f = 2y,$$

$$D_z f = 2z;$$

$$(D_x f)^2 + (D_y f)^2 + (D_z f)^2 = 4a^2.$$

Hence $\sigma = \int\int \frac{a}{z} dydx = a\int\int \frac{dydx}{\sqrt{a^2 - x^2 - y^2}}.$

Our limits of integration for y are $\sqrt{ax - x^2}$ and $\sqrt{a^2 - x^2}$; for x are 0 and a.

$$\sigma = a\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{dydx}{\sqrt{a^2 - x^2 - y^2}}.$$

$$\int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{dy}{\sqrt{a^2 - x^2 - y^2}} = \sin^{-1} \frac{y}{\sqrt{a^2 - x^2}}\Bigg]_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} = \frac{\pi}{2} - \sin^{-1}\sqrt{\frac{x}{a+x}}.$$

To find $\int_0^a \sin^{-1}\sqrt{\frac{x}{a+x}}.dx$ we must integrate by parts.

Let $u = \sin^{-1}\sqrt{\frac{x}{a+x}},$

and $dv = dx;$

$$v = x,$$

$$du = \frac{1}{2(a+x)}\sqrt{\frac{a}{x}}.dx;$$

$$\int \sin^{-1}\sqrt{\frac{x}{a+x}}.dx = x\sin^{-1}\sqrt{\frac{x}{a+x}} - \frac{\sqrt{a}}{2}\int\frac{\sqrt{x}}{a+x}.dx.$$

Let $$w = \sqrt{x}; \quad 2\,w dw = dx$$

and $$\int \frac{\sqrt{x}.dx}{a+x} = 2\int \frac{w^2 dw}{a+w^2} = 2\int\left(1 - \frac{a}{a+w^2}\right)dw.$$

$$\int \frac{\sqrt{x}.dx}{a+x} = 2\left(w - \sqrt{a}\tan^{-1}\frac{w}{\sqrt{a}}\right),$$

$$\int_0^a \sin^{-1}\sqrt{\frac{x}{a+x}}.dx$$

$$= a\sin^{-1}\frac{\sqrt{2}}{2} + a\tan^{-1}1 - a = \frac{a\pi}{4} + \frac{a\pi}{4} - a = \frac{a\pi}{2} - a,$$

$$\sigma = a\left(\frac{a\pi}{2} - \frac{a\pi}{2} + a\right) = a^2.$$

$8\sigma = 8a^2$ is the whole surface in question.

EXAMPLES.

(1) Find the area included by the cylinders described in Art. 133 by direct integration.

(2) A square hole is cut through a sphere, the axis of the hole coinciding with a diameter of the sphere; find the area of the surface removed.

(3) A cylinder is constructed on a single loop of the curve $r = a\cos n\theta$, having its generating lines perpendicular to the plane of this curve; determine the area of the portion of the surface of the sphere $x^2 + y^2 + z^2 = a^2$ which the cylinder intercepts.

Ans. $\dfrac{4a^2}{n}\left(\dfrac{\pi}{2} - 1\right)$.

(4) The centre of a regular hexagon moves along a diameter of a given circle (radius $= a$), the plane of the hexagon being perpendicular to this diameter, and its magnitude varying in such a manner that one of its diagonals always coincides with a chord of the circle; find the surface generated.

Ans. $a^2(2\pi + 3\sqrt{3})$.

CHAPTER XII.

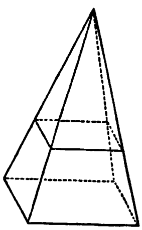
VOLUMES.

Single Integration.

134. If sections of a solid are made by parallel planes, and a set of cylinders drawn, each having for its base one of the sections, and for its altitude the distance between two adjacent cutting planes, the limit of the sum of the volumes of these cylinders, as the distance between the sections is indefinitely decreased, is the volume of the solid.

We shall take as established by Geometry the fact that the volume of a cylinder or prism is the product of the area of its base by its altitude.

It follows from what has just been said, that if, in a given solid, all of a set of parallel sections are equal, the volume of the solid is its base by its altitude, no matter how irregular its form.



Let us find the volume of a pyramid having b for the area of its base, and a for its altitude.

Divide the pyramid by planes parallel to the base, and let z be the area of a section at the distance x from the vertex.

We know from Geometry that $\frac{z}{b} = \frac{x^2}{a^2}$.

Hence $$z = \frac{b}{a^2}x^2.$$

Let the distance between two adjacent sections be dx; then the volume of the cylinder on z is

$$\frac{b}{a^2}x^2dx,$$

and V, the required volume of the pyramid, is

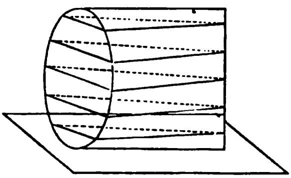
$$V = \frac{b}{a^2}\int_0^a x^2dx = \frac{ab}{3}.$$

Precisely the same reasoning applies to any cone, which will therefore have for its volume one-third the product of its base by its altitude.

EXAMPLE.

Find the volume of the frustum of a pyramid or of a cone.

135. If a line move keeping always parallel to a given plane, and touching a plane curve and a straight line parallel to the plane of the curve, the surface generated is called a *conoid*. Let us find the volume of a *conoid* when the director line and curve are perpendicular to the given plane.



Divide the conoid into laminae by planes parallel to the fixed plane.

Let Δy be the distance between two adjacent sections, and let x be the length of the line in which any section cuts the base of the conoid; let a be the altitude and b the area of the base of the figure. Any one of our elementary cylinders will have for its volume $\frac{1}{2}ax\Delta y$, since the area of its triangular base is $\frac{1}{2}ax$, and we have $V = \frac{1}{2}a\int xdy$, the limits of integration being so taken as to embrace the whole solid. $\int xdy$ between the limits in question is the area of the base of the conoid; hence its volume,

$$V = \tfrac{1}{2}ab.$$

EXAMPLES.

(1) Find the volume of a conoid when the director line and curve are not perpendicular to the given plane.

(2) A woodman fells a tree 2 feet in diameter, cutting halfway through from each side. The lower face of each cut is horizontal, and the upper face makes an angle of 45° with the horizontal. How much wood does he cut out?

136. To find the volume of an ellipsoid.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Take the cutting planes parallel to the plane of XY. A section at the distance z from the origin will have

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{z^2}{c^2}=\frac{c^2-z^2}{c^2}$$

for its equation, and $\frac{a}{c}\sqrt{c^2-z^2}$ and $\frac{b}{c}\sqrt{c^2-z^2}$ for its semi-axes; hence its area will be $\frac{\pi ab}{c^2}(c^2-z^2)$.

Any of the elementary cylinders will have for its volume $\frac{\pi ab}{c^2}(c^2-z^2)\Delta z$, and we shall have for the whole solid

$$V=\frac{\pi ab}{c^2}\int_{-c}^{c}(c^2-z^2)\,dz.$$

$$V=\tfrac{4}{3}\pi abc.$$

If a, b, and c are equal, the ellipsoid is a sphere, and

$$V=\tfrac{4}{3}\pi a^3.$$

EXAMPLES.

(1) Find the volume included between an hyperboloid of one sheet

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,$$

and its asymptotic cone

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0.$$

Ans. It is equal to a cylinder of the same altitude as the solid in question, and having for a base the section made by the plane of XY.

(2) Find the whole volume of the solid bounded by the surface

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^4}{c^4}=1. \qquad \textit{Ans. } \frac{8\pi abc}{5}.$$

(3) Find the volume cut from the surface

$$\frac{z^2}{c}+\frac{y^2}{b}=2x$$

by a plane parallel to the plane of (YZ) at a distance a from it.

Ans. $\pi a^2\sqrt{(bc)}$.

(4) Find the whole volume of the solid bounded by

$$(x^2+y^2+z^2)^3=27\,a^3xyz. \qquad \textit{Ans. } \tfrac{9}{2}a^3.$$

(5) The centre of a regular hexagon moves along a diameter of a given circle (radius $=a$), the plane of the hexagon being perpendicular to this diameter, and its magnitude varying in such a manner that one of its diagonals always coincides with a chord of the circle; find the volume generated.

Ans. $2\sqrt{3}.a^3$.

(6) A circle (radius $=a$) moves with its centre on the circumference of an equal circle, and keeps parallel to a given plane which is perpendicular to the plane of the given circle; find the volume of the solid it will generate.

Ans. $\dfrac{2a^3}{3}(3\pi+8)$.

Solids of Revolution. Single Integration.

137. If a solid is generated by the revolution of a plane curve $y=fx$ about the axis of x, sections made by planes perpendicular to the axis are circles. The area of any such circle is πy^2, the volume of the elementary cylinder is $\pi y^2\Delta x$, and

$$V=\pi\int_{x_0}^{x}y^2dx$$

is the volume of the solid generated.

For example; let us find the volume of the solid generated by the revolution of one branch of the tractrix about the axis of X. Here we must integrate from $x=0$ to $x=\infty$.

$$V=\pi\int_0^{\infty}y^2\,dx.$$

We have $$dx = -\frac{(a^2 - y^2)^{\frac{1}{2}}}{y}dy \qquad \text{Art. 91 [2]}$$

in the case of the tractrix;

hence $$V = -\pi\int_{x=0}^{x=\infty} y(a^2 - y^2)^{\frac{1}{2}} dy.$$

When $x = 0$, $y = a$, and when $x = \infty$, $y = 0$.

Therefore $$V = -\pi\int_a^0 y(a^2 - y^2)^{\frac{1}{2}} dy = \frac{\pi a^3}{3}.$$

EXAMPLES.

(1) If the plane curve revolves about the axis of Y,

$$V = \pi\int_{y_0}^{y_1} x^2\,dy.$$

(2) The volume of a sphere is $\frac{4}{3}\pi a^3$.

(3) The volume of the solid formed by the revolution of a cycloid about its base is $5\pi^2 a^3$.

(4) The curve $y^2(2a - x) = x^3$ revolves about its asymptote; show that the volume generated is $2\pi^2 a^3$.

Solids of Revolution. Double Integration.

138. If we suppose the area of the revolving curve broken up into infinitesimal rectangles as in Art. 125, the element $\Delta x \Delta y$ at any point P, whose coördinates are x and y, will generate a ring the volume of which will differ from $2\pi y \Delta x \Delta y$ by an amount which will be an infinitesimal of higher order than the second if we regard Δx and Δy as of the first order. For the ring in question is obviously greater than a prism having the same cross-section $\Delta x \Delta y$, and having an altitude equal to the inner circumference $2\pi y$ of the ring, and is less than a prism having $\Delta x \Delta y$ for its base and $2\pi(y + \Delta y)$, the outer circumference of the ring, for its altitude; but these two prisms differ by $2\pi \Delta x(\Delta y)^2$, which is of the third order.

$\Delta x \int_0^y 2\pi y dy$, where the upper limit of integration is the ordinate of the point of the curve immediately above P, and must be expressed in terms of x by the aid of the equation of the revolving curve, will give us the elementary cylinder used in Art. 137.

The whole volume required will be the limit of the sum of these cylinders; that is,

$$V = 2\pi \int_{x_0}^{x_1} \int_0^y y dy dx. \qquad [1]$$

If the figure revolved is bounded by two curves, the required volume can be found by the formula just obtained, if the limits of integration are suitably chosen.

Let us consider the following example:

A paraboloid of revolution has its axis coincident with the diameter of a sphere, and its vertex in the surface of the sphere; required the volume between the two surfaces.

Let $$y^2 = 2mx \qquad (1)$$

be the parabola, and $$x^2 + y^2 - 2ax = 0 \qquad (2)$$

be the circle, which form the paraboloid and the sphere by their revolution. The abscissas of their points of intersection are 0 and $2(a - m)$.

We have $$V = 2\pi \int \int y dy dx,$$

and, in performing our first integration, our limits must be the values of y obtained from equations (1) and (2).

We get $$V = \pi \int [2(a - m)x - x^2] dx,$$

and here our limits of integration are 0 and $2(a - m)$.

Hence $$V = \tfrac{4}{3}\pi(a - m)^3 = \frac{\pi h^3}{6},$$

if h is the altitude of the solid in question.

EXAMPLES.

(1) A cone of revolution and a paraboloid of revolution have the same vertex and the same base; required the volume between them. *Ans.* $\frac{\pi m h^2}{3}$, where h is the altitude of the cone.

(2) Find the volume included between a right cone, whose vertical angle is 60°, and a sphere of given radius touching it along a circle.

Ans. $\dfrac{\pi r^3}{6}$.

Solids of Revolution. Polar Formula.

139. If we use polar coördinates, and suppose the revolving area broken up, as in Art. 127, into elements of which $rd\phi dr$ is the one at any point P whose coördinates are r and ϕ, the element $rd\phi dr$ will generate a ring whose volume will differ from $2\pi r^2 \sin\phi d\phi dr$ by an infinitesimal of higher order than the second, if we regard $d\phi$ and dr as of the first order; for it will be less than a prism having for its base $rd\phi dr$, and for its altitude $2\pi(r+dr)\sin(\phi+d\phi)$, and greater than a prism having the same base and the altitude $2\pi r\sin\phi$; and these prisms differ by an amount which is infinitesimal of higher order than the second.

We shall have then

$$V = 2\pi\int\int r^2 \sin\phi dr d\phi, \qquad [1]$$

the limits being so taken as to bring in the whole of the generating area.

For example; let us find the volume generated by the revolution of a cardioide about its axis.

$$r = 2a(1-\cos\phi)$$

is the equation of the cardioide;

$$V = 2\pi\int\int r^2 \sin\phi dr d\phi.$$

Our first integral must be taken between the limits $r=0$ and $r=2a(1-\cos\phi)$, and is

$$\frac{8a^3}{3}(1-\cos\phi)^3\sin\phi d\phi.$$

$$V = \frac{16}{3}a^3\pi\int_0^\pi (1-\cos\phi)^3\sin\phi d\phi,$$

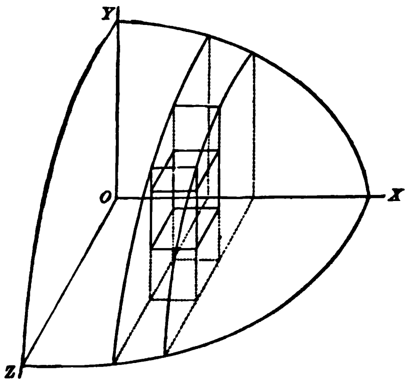
$$V = \frac{64}{3}\pi a^3.$$

EXAMPLE.

A right cone has its vertex on the surface of a sphere, and its axis coincident with the diameter of the sphere passing through that point; find the volume common to the cone and the sphere.

Volume of any Solid. Triple Integration.

140. If we suppose our solid divided into parallelopipeds by planes parallel to the three coördinate planes, the elementary



parallelopiped at any point (x,y,z) within the solid will have for its volume $\Delta x \Delta y \Delta z$, or, if we regard x, y, and z as independent, $dxdydz$; and the whole volume

$$V = \int\int\int dxdydz, \qquad [1]$$

the limits being so chosen as to embrace the whole solid.

The integrations are independent, and may be performed in any order if the limits are suitably chosen.

For example; let us find the volume of the portion of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

cut off by the coördinate planes.

$$V=\int\int\int dzdydx,$$

and our limits are, for z, 0 and $c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$; for y, 0 and $b\sqrt{1-\frac{x^2}{a^2}}$; and for x, 0 and a. For, starting at any point (x,y,z) and integrating on the hypothesis that z alone varies, we get a column of our elementary parallelopipeds having $dxdy$ as a base and passing through the point (x,y,z). To make this column reach from the plane XY to the surface, z must increase from the value zero to the value belonging to the point on the surface of the ellipsoid which has the coördinates x and y; that is, to the value $c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$. Then, integrating on the hypothesis that y alone varies, we shall sum these columns and shall get a slice of the solid passing through (x,y,z) and having the thickness dx. To make this slice reach completely across the solid, we must let y increase from the value zero to the greatest value it can have in the slice in question; that is, to the value which is the ordinate of that point of the section of the ellipsoid by the plane XY which has the abscissa x. The section in question has the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1;$$

therefore the required value of y is $b\sqrt{1-\frac{x^2}{a^2}}$.

Last, in integrating on the hypothesis that x alone varies, we must choose our limits so as to include all the slices just described, and must increase x from zero to a.

$$\int dz=z=c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$$

between the limits 0 and $c^2\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$

$$c\int\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}.dy$$

$$=\frac{c}{b}\int\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}.dy$$

$$=\tfrac{1}{2}\frac{c}{b}\left[y\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}+b^2\left(1-\frac{x^2}{a^2}\right)\sin^{-1}\frac{y}{b\sqrt{1-\frac{x^2}{a^2}}}\right]$$

$$=\frac{\pi bc}{4}\left(1-\frac{x^2}{a^2}\right)$$

between the limits 0 and $b\sqrt{1-\frac{x^2}{a^2}}$.

$$\frac{\pi bc}{4}\int_0^a\left(1-\frac{x^2}{a^2}\right)dx=\frac{\pi abc}{6},$$

the volume required.

EXAMPLES.

(1) Find the volume obtained in the present article, performing the integrations in the order indicated by the formula,

$$V=\int\int\int dxdydz.$$

(2) Find the volume cut off from the surface

$$\frac{z^2}{c}+\frac{y^2}{b}=2x$$

by a plane parallel to that of YZ, at a distance a from it.

Ans. $\pi a^2\sqrt{(bc)}$.

(3) Find the volume enclosed by the surfaces,

$$x^2+y^2=cz,\quad x^2+y^2=ax,\quad z=0.$$

Ans. $\frac{3\pi a^4}{32c}$.

(4) Obtain the volume bounded by the surface

$$z=a-\sqrt{x^2+y^2}$$

and the planes $x=z$ and $x=0$.

Ans. $\frac{2a^3}{9}$.

5. Find the volume of the conoid bounded by the surface $z^2+\dfrac{a^2y^2}{x^2}=c^2$ and the planes $x=0$ and $x=a$. *Ans.* $\dfrac{\pi c^2 a}{2}$.

141. If we use polar coördinates we can take as our element of volume

$$r^2 \sin\phi\, dr\, d\phi\, d\theta,$$

an expression easily obtained from the element $2\pi r^2 \sin\phi\, dr\, d\phi$ used in Art. 139.

Then $$V=\int\int\int r^2 \sin\phi\, dr\, d\phi\, d\theta,$$

where the order of the integrations is usually immaterial if the limits are properly chosen.

Example.

Find the volume of a sphere by polar coördinates.

CHAPTER XIII.

CENTRES OF GRAVITY.

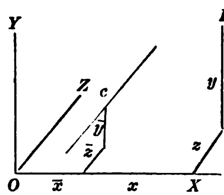
142. The *moment of a force about an axis* is the product of the magnitude of the force by the perpendicular distance of its line of direction from the axis, and measures the tendency of the force to produce rotation about the axis.

The *force exerted by gravity* on any material body is proportional to the mass of the body, and may be measured by the mass of the body.

The *Centre of Gravity* of a body is a point so situated that the force of gravity produces no tendency in the body to rotate about any axis passing through this point.

The subject of centres of gravity belongs to Mechanics, and we shall accept the definitions and principles just stated as data for mathematical work, without investigating the mechanical grounds on which they rest.

143. Suppose the points of a body referred to a set of three rectangular axes fixed in the body, and let $\bar{x},\bar{y},\bar{z}$ be the coördinates of the centre of gravity. Place the body with the axes of X and Z horizontal, and consider the tendency of the particles of the body to produce rotation about an axis through $(\bar{x},\bar{y},\bar{z})$ parallel to OZ, under the influence of gravity. Represent the mass of an elementary parallelopiped at any point (x,y,z) by dm. The force exerted by gravity on dm is measured by dm, and its line of direction is vertical. If the mass of dm were concentrated at P, the moment of the force exerted on dm about the



axis through C would be $(x-\bar{x})dm$, and this moment would represent the tendency of dm to rotate about the axis in question; the tendency of the whole body to rotate about this axis would be $\Sigma(x-\bar{x})dm$. If now we decrease dm indefinitely, the error committed in assuming that the mass of dm is concentrated at P decreases indefinitely, and we shall have as the true expression for the tendency of the whole body to rotate about the axis through C, $\int(x-\bar{x})dm$; but this must be zero.

Hence
$$\int(x-\bar{x})dm=0,$$
$$\int xdm-\bar{x}\int dm=0,$$
$$\bar{x}=\frac{\int xdm}{\int dm}. \qquad [1]$$

If we place the body so that the axes of Y and X are horizontal, the same reasoning will give us

$$\bar{y}=\frac{\int ydm}{\int dm}; \qquad [2]$$

and in like manner we can get

$$\bar{z}=\frac{\int zdm}{\int dm}. \qquad [3]$$

Since $\int dm$ is the mass of the whole body, if we represent it by M we shall have

$$\bar{x}=\frac{\int xdm}{M},$$
$$\bar{y}=\frac{\int ydm}{M},$$
$$\bar{z}=\frac{\int zdm}{M}.$$

EXAMPLE.

Show that the effect of gravity in making a body tend to rotate any given axis is precisely the same as if the mass of the body were concentrated at its centre of gravity.

144. The mass of any homogeneous body is the product of its volume by its density. If the body is not homogeneous, the density at any point will be a function of the position of that point. Let us represent it by κ. Then we may regard dm as equal to κdv if dv is the element of volume, and we shall have

$$\bar{x} = \frac{\int x\kappa dv}{\int \kappa dv} \qquad [1]$$

and corresponding formulas for $\bar{y}$ and $\bar{z}$.

If the body considered is homogeneous, κ is constant, and we shall have

$$\bar{x} = \frac{\int xdv}{\int dv} = \frac{\int xdv}{V}, \qquad [2]$$

$$\bar{y} = \frac{\int ydv}{\int dv} = \frac{\int ydv}{V}, \qquad [3]$$

$$\bar{z} = \frac{\int zdv}{\int dv} = \frac{\int zdv}{V}. \qquad [4]$$

In any particular problem we have only to express dv in terms of the coördinates.

Plane Area.

145. If we use rectangular coördinates, and are dealing with a plane area, where the weight is uniformly distributed, we have

$$dv = dA = dxdy. \qquad \text{(Art. 125).}$$

Hence, by 144, [2] and [3],

$$\left.\begin{aligned}\bar{x}&=\frac{\int\int x\,dx\,dy}{\int\int dx\,dy}\\ \bar{y}&=\frac{\int\int y\,dx\,dy}{\int\int dx\,dy}\end{aligned}\right\}. \qquad [1]$$

If we use polar coördinates,

$$dv = dA = r\,d\phi\,dr,$$

and

$$\left.\begin{aligned}\bar{x}&=\frac{\int\int r^2\cos\phi\;d\phi\,dr}{\int\int r\,d\phi\,dr}\\ \bar{y}&=\frac{\int\int r^2\sin\phi\;d\phi\,dr}{\int\int r\,d\phi\,dr}\end{aligned}\right\}. \qquad [2]$$

For example; let us find the *centre of gravity* of the area between the *cissoid* and its asymptote. From the equation of the cissoid

$$y^2 = \frac{x^3}{a-x},$$

we see that the curve is symmetrical with respect to the axis of X, passes through the origin, and has the line $x = a$ as an asymptote. From the symmetry of the area in question, $\bar{y} = 0$, and we need only find $\bar{x}$.

$$\bar{x} = \frac{\int_0^a\int_{-y}^{y} x\,dy\,dx}{\int_0^a\int_{-y}^{y} dy\,dx} = \frac{\int_0^a xy\,dx}{\int_0^a y\,dx},$$

$$\bar{x}=\frac{\displaystyle\int_0^a\frac{x^{\frac{5}{2}}}{(a-x)^{\frac{1}{2}}}dx}{\displaystyle\int_0^a\frac{x^{\frac{1}{2}}}{(a-x)^{\frac{1}{2}}}dx}=\frac{\displaystyle\tfrac{5}{6}a\int_0^a\frac{x^{\frac{3}{2}}}{(a-x)^{\frac{1}{2}}}dx}{\displaystyle\int_0^a\frac{x^{\frac{3}{2}}}{(a-x)^{\frac{1}{2}}}dx};\quad\text{by Art. 64 [4].}$$

$$\bar{x}=\tfrac{5}{6}a.$$

As an example of the use of the polar formulas [2], let us find the *centre of gravity* of the cardioide

$$r=2a(1-\cos\phi).$$

Here, from the fact that the axis of X is an axis of symmetry, we know that $\bar{y}=0$.

$$\bar{x}=\frac{\displaystyle\int_0^{2\pi}\int_0^r r^2\cos\phi\, dr\, d\phi}{\displaystyle\int_0^{2\pi}\int_0^r r\, dr\, d\phi}$$

$$=\frac{\displaystyle\tfrac{1}{3}\int_0^{2\pi}r^3\cos\phi\, d\phi}{\displaystyle\tfrac{1}{2}\int_0^{2\pi}r^2\, d\phi}=\frac{\displaystyle\frac{8a^3}{3}\int_0^{2\pi}(1-\cos\phi)^3\cos\phi\, d\phi}{\displaystyle 2a^2\int_0^{2\pi}(1-\cos\phi)^2\, d\phi},$$

$$\int_0^{2\pi}(\cos\phi-3\cos^2\phi+3\cos^3\phi-\cos^4\phi)\,d\phi=-\tfrac{15}{4}\pi;$$

and $$\int_0^{2\pi}(1-2\cos\phi+\cos^2\phi)\,d\phi=3\pi.$$

Hence $$\bar{x}=-\tfrac{5}{3}a.$$

EXAMPLES.

1. Show that formulas [1] hold even when we use *oblique coördinates*.

2. Find the centre of gravity of a segment of a parabola cut off by any chord.

Ans. $\bar{x}=\frac{3}{5}a$, $\bar{y}=0$. If the axes are the tangent parallel to the chord and the diameter bisecting the chord.

3. Find the centre of gravity of the area bounded by the semi-cubical parabola $ay^2 = x^3$ and a double ordinate. *Ans.* $\bar{x} = \frac{5}{7}x$.

4. Find the centre of gravity of a semi-ellipse, the bisecting line being any diameter.

Ans. If the bisecting diameter is taken as the axis of Y, and the conjugate diameter as the axis of X, $\bar{x} = \frac{4a}{3\pi}$, $\bar{y} = 0$.

5. Find the centre of gravity of the curve $y^2 = b^2 \frac{a-x}{x}$.

Ans. $\bar{x} = \frac{1}{4}a$.

6. Find the centre of gravity of the cycloid.

Ans. $\bar{x} = a\pi$, $\bar{y} = \frac{5}{6}a$.

7. Find the centre of gravity of the lemniscate $r^2 = a^2 \cos 2\phi$.

Ans. $\bar{x} = \frac{\pi\sqrt{2}}{8}a$.

8. Find the centre of gravity of a circular sector.

Ans. If we take the radius bisecting the sector as the axis of X, and represent the angle of the sector by 2α, $\bar{x} = \frac{2}{3}\frac{a \sin\alpha}{\alpha}$.

9. Find the centre of gravity of the segment of an ellipse cut off by a quadrantal chord. *Ans.* $\bar{x} = \frac{2}{3}\frac{a}{\pi - 2}$, $\bar{y} = \frac{2}{3}\frac{b}{\pi - 2}$.

10. Find the centre of gravity of a quadrant of the area of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. *Ans.* $\bar{x} = \bar{y} = \frac{256}{315}\frac{a}{\pi}$.

146. If we are dealing with a homogeneous solid formed by the revolution of a plane curve about the axis of X, we have

$$dv = 2\pi y dy dx. \qquad \text{(Art. 138 [1])}$$

Hence, by Art. 144 [2],

$$\bar{x} = \frac{\int\int xy dx dy}{\int\int y dx dy}. \qquad [1]$$

If we use polar coördinates,

$$dv = 2\pi r^2 \sin\phi dr d\phi. \qquad \text{(Art. 139 [1])}$$

Hence

$$\bar{x} = \frac{\int\int r^3 \sin\phi \cos\phi dr d\phi}{\int\int r^2 \sin\phi dr d\phi}. \qquad [2]$$

For example; let us find the centre of gravity of a hemisphere. The equation of the revolving curve is $x^2 + y^2 = a^2$.

$$\bar{x} = \frac{\int_0^a \int_0^{\sqrt{a^2-x^2}} xy dy dx}{\int_0^a \int_0^{\sqrt{a^2-x^2}} y dy dx} = \frac{\frac{1}{8}a^4}{\frac{1}{3}a^3} = \frac{3}{8}a.$$

If we use polar coördinates the equation of the revolving curve is $r = a$.

Here

$$\bar{x} = \frac{\int_0^a \int_0^{\frac{\pi}{2}} r^3 \sin\phi \cos\phi d\phi dr}{\int_0^a \int_0^{\frac{\pi}{2}} r^2 \sin\phi d\phi dr} = \frac{\frac{1}{8}a^4}{\frac{1}{3}a^3} = \frac{3}{8}a.$$

EXAMPLES.

1. Find the centre of gravity of the solid formed by the revolution of the sector of a circle about one of its extreme radii.

Ans. $\bar{x} = \frac{3}{4}a\cos^2\frac{1}{2}\beta$, where β is the angle of the sector.

2. Find the centre of gravity of the segment of a paraboloid of revolution cut off by a plane perpendicular to the axis.

Ans. $\bar{x} = \frac{2}{3}a$, where $x = a$ is the plane.

3. Find the centre of gravity of the solid formed by scooping out a cone from a given paraboloid of revolution, the bases of the two volumes being coincident as well as their vertices.

Ans. The centre of gravity bisects the axis.

4. A cardioide is made to revolve about its axis; find the centre of gravity of the solid generated. *Ans.* $\bar{x} = -\frac{4}{5}a$.

5. Obtain formulas for the centre of gravity of any homogeneous solid.

6. Find the centre of gravity of the solid bounded by the surface $z^2 = xy$ and the five planes $x=0$, $y=0$, $z=0$, $x=a$, $y=b$.
Ans. $\bar{x} = \frac{3}{5}a$, $\bar{y} = \frac{3}{5}b$, $\bar{z} = \frac{9}{32}a^{\frac{1}{2}}b^{\frac{1}{2}}$.

147. If we are dealing with the arc of a plane curve, the formulas of Art. 144 reduce to

$$\bar{x} = \frac{\int x\,ds}{\int ds}, \qquad [1]$$

$$\bar{y} = \frac{\int y\,ds}{\int ds}. \qquad [2]$$

EXAMPLES.

1. Find the centre of gravity of an arc of a circle, taking the diameter bisecting the arc as the axis of X and the centre as the origin. *Ans.* $\bar{x} = \frac{ac}{s}$, where c is the chord of the arc.

2. Find the centre of gravity of the arc of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ between two successive cusps. *Ans.* $\bar{x} = \bar{y} = \frac{2}{5}a$.

3. Find the centre of gravity of the arc of a semi-cycloid.
Ans. $\bar{x} = (\pi - \frac{4}{3})a$, $\bar{y} = -\frac{2}{3}a$.

4. Find the centre of gravity of the arc of a catenary cut off by any double ordinate.
Ans. $\bar{x} = 0$, $\bar{y} = \frac{ax + sy}{2s}$, where $2s$ is the length of the arc.

5. Obtain formulas for the centre of gravity of a surface of revolution, the weight being uniformly distributed over the surface.

6. Find the centre of gravity of any zone of a sphere.

Ans. The centre of gravity bisects the line joining the centres of the bases of the zone.

7. A cardioide revolves about its axis; find the centre of gravity of the surface generated. *Ans.* $\bar{x} = -\frac{100}{63}a$.

8. Find the centre of gravity of the surface of a hemisphere when the density at each point of the surface varies as its perpendicular distance from the base of the hemisphere.

Ans. $\bar{x} = \frac{2}{3}a$.

9. Find the centre of gravity of a quadrant of a circle, the density at any point of which varies as the *nth* power of its distance from the centre. *Ans.* $\bar{x} = \bar{y} = \frac{n+2}{n+3} \cdot \frac{2a}{\pi}$.

10. Find the centre of gravity of a hemisphere, the density of which varies as the distance from the centre of the sphere.

Ans. $\bar{x} = \frac{2}{5}a$.

Properties of Guldin.

148. I. If a plane area revolve about an axis external to itself through any assigned angle, the volume of the solid generated will be equal to a prism whose base is the revolving area and whose altitude is the length of the path described by the centre of gravity of the area.

II. If the arc of a plane curve revolve about an external axis in its own plane through any assigned angle, the area of the surface generated will be equal to that of a rectangle, one side of which is the length of the revolving curve, and the other the length of the path described by its centre of gravity.

First; let the area in question revolve about the axis of X through an angle Θ. The ordinate of the centre of gravity of the area in question is

$$\bar{y} = \frac{\int\int y\,dx\,dy}{\int\int dx\,dy}, \qquad \text{by Art. 145 [1].}$$

The length of the path described by the centre of gravity

$$\bar{y}\Theta = \frac{\Theta\int\int y\,dx\,dy}{\int\int dx\,dy}. \qquad (1)$$

The volume generated is

$$V = \Theta\int\int y\,dx\,dy, \qquad \text{by Art. 138.}$$

Hence
$$V = \bar{y}\Theta\int\int dx\,dy.$$

But $\int\int dx\,dy$ is the revolving area, and the first theorem is established.

We leave the proof of the second theorem to the student.

EXAMPLES.

1. Find the surface and volume of a sphere, regarding it as generated by the revolution of a semicircle.

2. Find the surface and volume of the solid generated by the revolution of a cycloid about its base.

3. Find the volume and the surface of the ring generated by the revolution of a circle about an external axis.

Ans. $V = 2\pi^2a^2b$, $S = 4\pi^2ab$, where b is the distance of the centre of the circle from the axis.

4. Find the volume of the ring generated by the revolution of an ellipse about an external axis.

Ans. $V = 2\pi^2abc$, where c is the distance of the centre of the ellipse from the axis.

CHAPTER XIV.

MEAN VALUE AND PROBABILITY.

149. The application of the Integral Calculus to questions in Mean Value and Probability is a matter of decided interest; but lack of space will prevent our doing more than solving a few problems in illustration of some of the simplest of the methods and devices ordinarily employed. A full and admirable treatment of the subject is given in "Williamson's Integral Calculus" (London: Longmans, Green, & Co.); and numerous interesting problems are published with their solutions in "The Mathematical Visitor," a magazine edited by Artemas Martin, Erie, Pa.

150. The *mean* of n quantities is their *sum* divided by their *number*. If we are finding the *mean* value of a continuously-varying quantity, we have to consider an infinite number of values, and, of course, an infinite sum as well; a little ingenuity will enable us to throw the ratio of the sum to the number into a form to which we can apply the Integral Calculus.

(a) Let us find the mean distance of all the points on the circumference of a circle from a given point on the circumference.

If we take the given point as origin, the distances whose mean is required are the radii vectores of points uniformly distributed along the circumference of the circle.

Let the distance between any two adjacent points be ds; then n, the number of points considered, is equal to $\frac{2\pi a}{ds}$, if a is the

radius of the circle; and if r is the radius vector of any point of the circumference, and M the mean value required,

$$M = \frac{\Sigma r}{\frac{2\pi a}{ds}} = \frac{\Sigma r ds}{2\pi a}$$

for any finite value of ds. In the actual case under consideration,

$$M = \frac{\int r ds}{2\pi a}.$$

The polar equation of the circle is

$$r = 2a\cos\phi;$$

$$ds = 2a d\phi,$$

$$M = \frac{1}{2\pi a}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4a^2 \cos\phi \, d\phi = \frac{4a}{\pi},$$

the required *mean value*.

(*b*) Let us find the mean distance of points on the surface of a circle from a fixed point on the circumference.

Using the same notation as before, we shall have

$$n = \frac{\pi a^2}{r d\phi dr},$$

r and ϕ, the polar coördinates of any point of the surface, being independent variables.

$$M = \frac{\Sigma r}{\frac{\pi a^2}{r dr d\phi}} = \frac{\Sigma r^2 dr d\phi}{\pi a^2},$$

$$M = \frac{1}{\pi a^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2a\cos\phi} r^2 dr d\phi = \frac{32a}{9\pi},$$

the required *mean value*.

(*c*) As an example of a device often employed, we shall now solve the problem, To find the mean distance between two points within a given circle.

If M be the required mean, the sum of the whole number of cases can be represented by $(\pi r^2)^2 M$, r being the radius of the circle; since for each position of the first point the number of positions of the second point is proportional to the area of the circle, and may be measured by that area; and as the number of possible positions of the first point may also be measured by the area of the circle, the whole number of cases to be considered is represented by the square of the area; and the sum of all the distances to be considered must be the product of the mean distance by the number.

Let us see what change will be produced in this sum by increasing r by the infinitesimal dr; that is, let us find $d(\pi^2 r^4 M)$.

If the first point is anywhere on the annulus $2\pi r.dr$, which we have just added, its mean distance from the other points of the circle is $\dfrac{32r}{9\pi}$, by (*b*).

Therefore, the sum of the new distances to be considered, if the first point is on the annulus, is $\dfrac{32r}{9\pi}.\pi r^2.2\pi r dr$; but the second point may be on the annulus, instead of the first; so that to get the sum of all the new cases brought in by increasing r by dr, we must double the value just obtained.

Hence
$$d(\pi^2 r^4 M) = \tfrac{128}{9}\pi r^4 dr,$$
$$\pi^2 a^4 M = \tfrac{128}{9}\pi \int_0^a r^4 dr = \tfrac{128}{45}\pi a^5,$$
$$M = \frac{128a}{45\pi}.$$

151. In solving questions in *Probability*, we shall assume that the student is familiar with the elements of the theory as given in "Todhunter's Algebra."

(*a*) A man starts from the bank of a straight river, and walks till noon in a random direction; he then turns and walks

in another random direction; what is the probability that he will reach the river by night?

Let θ be the angle his first course makes with the river. If the angle through which he turns at noon is less than $\pi - 2\theta$, he will reach the river by night. For any given value of θ, then, the required probability is $\frac{\pi - 2\theta}{2\pi}$. The probability that θ shall lie between any given value θ_0 and $\theta_0 + d\theta$ is $\frac{d\theta}{\frac{1}{2}\pi}$.

The chance that his first course shall make an angle with the river between θ_0 and $\theta_0 + d\theta$, and that he shall get back, is

$$\frac{\pi - 2\theta}{2\pi} \cdot \frac{d\theta}{\frac{1}{2}\pi} = \frac{(\pi - 2\theta)\,d\theta}{\pi^2}.$$

As θ is equally likely to have any value between 0 and $\frac{\pi}{2}$, the required probability,

$$p = \int_0^{\frac{1}{2}\pi} \frac{(\pi - 2\theta)\,d\theta}{\pi^2} = \tfrac{1}{4}.$$

(*b*) A floor is ruled with equidistant straight lines; a rod, shorter than the distance between the lines, is thrown at random on the floor; to find the chance of its falling on one of the lines.

Let x be the distance of the centre of the rod from the nearest line; θ the inclination of the rod to a perpendicular to the parallels passing through the centre of the rod; $2a$ the common distance of the parallels; $2c$ the length of the rod.

In order that the rod may cross a line, we must have $c\cos\theta > x$; the chance of this for any given value x_0 of x is $\frac{1}{\frac{1}{2}\pi}\cos^{-1}\frac{x_0}{c}$.

The probability that x will have the value x_0 is $\frac{dx}{a}$. The probability required is

$$p = \frac{2}{\pi a}\int_0^c \cos^{-1}\frac{x}{c}\,dx = \frac{2c}{\pi a}.$$

This problem may be solved by another method which possesses considerable interest.

Since all values of x from 0 to a, and all values of θ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ are equally probable, the whole number of cases that can arise may be represented by

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_0^a dx d\theta = \pi a.$$

The number of favorable cases will be represented by

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_0^{c\cos\theta} dx d\theta = 2c.$$

Hence $$p = \frac{2c}{\pi a}.$$

(c) To find the probability that the distance of two stars, taken at random in the northern hemisphere, shall exceed 90°.

Let α be the latitude of the first star. With the star as a pole, describe an arc of a great circle, dividing the hemisphere into two lunes; the probability that the distance of the second star from the first will exceed 90° is the ratio of the lune not containing the first star to the hemisphere, and is equal to $\frac{(\frac{1}{2}\pi - \alpha)}{\pi}$. The probability that the latitude of the first star will be between α and $\alpha + d\alpha$ is the ratio of the area of the zone, whose bounding circles have the latitudes α and $\alpha + d\alpha$ respectively, to the area of the hemisphere, and is

$$\frac{2\pi a^2 \cos\alpha\, d\alpha}{2\pi a^2} = \cos\alpha\, d\alpha.$$

Hence $$p = \int_0^{\frac{\pi}{2}} \frac{(\frac{1}{2}\pi - \alpha)}{\pi} \cos\alpha\, d\alpha = \frac{1}{\pi}.$$

(d) A random straight line meets a closed convex curve; what is the probability that it will meet a second closed convex curve within the first?

If an infinite number of random lines be drawn in a plane, all directions are equally probable; and lines having any given

direction will be disposed with equal frequency all over the plane. If we determine a line by its distance p from the origin, and by the angle α which p makes with the axis of X, we can get all the lines to be considered by making p and α vary between suitable limits by equal infinitesimal increments.

In our problem, the whole number of lines meeting the external curve can be represented by $\int\int dpd\alpha$. If the origin is within the curve, the limits for p must be zero, and the perpendicular distance from the origin to a tangent to the curve; and for α must be zero and 2π. If we call this number N, we shall have

$$N = \int_0^{2\pi} pd\alpha,$$

p being now the perpendicular from the origin to the tangent.

If we regard the distance from a given point of any closed convex curve along the curve to the point of contact of a tangent, and then along the tangent to the foot of the perpendicular let fall upon it from the origin, as a function of the α used above, its differential is easily seen to be $pd\alpha$. If we sum these differentials from $\alpha = 0$ to $\alpha = 2\pi$, we shall get the perimeter of the given curve.

Hence $$N = \int_0^{2\pi} pd\alpha = L,$$

where L is the perimeter of the curve in question. By the same reasoning, we can see that n, the number of the random lines which meet the inner curve, is equal to l, its perimeter. For p, the required probability, we shall have

$$p = \frac{l}{L}.$$

EXAMPLES.

(1) A number n is divided at random into two parts; find the mean value of their product. *Ans.* $\frac{n^2}{6}$.

(2) Find the mean value of the ordinates of a semicircle, supposing the series of ordinates taken equidistant. *Ans.* $\frac{\pi}{4}a$.

(3) Find the mean value of the ordinates of a semicircle, supposing the ordinates drawn through equidistant points on the circumference. *Ans.* $\frac{2a}{\pi}$.

(4) Find the mean values of the roots of the quadratic $x^2 - ax + b = 0$, the roots being known to be real, but b being unknown but positive. *Ans.* $\frac{5a}{6}$ and $\frac{a}{6}$.

(5) Prove that the mean of the radii vectores of an ellipse, the focus being the origin, is equal to half the minor axis when they are drawn at equal angular intervals, and is equal to half the major axis when they are drawn so that the abscissas of their extremities increase uniformly.

(6) Suppose a straight line divided at random into three parts; find the mean value of their product. *Ans.* $\frac{a^3}{60}$.

(7) Find the mean square of the distance of a point within a given square (side $= 2a$) from the centre of the square. *Ans.* $\frac{2}{3}a^2$.

(8) A slab is sawed at random from a round log, find its mean thickness. *Ans.* $\frac{4a}{3\pi}$.

(9) A chord is drawn joining two points taken at random on a circle; find the mean area of the less of the two segments into which it divides the circle. *Ans.* $\frac{\pi a^2}{4} - \frac{a^2}{\pi}$.

(10) Find the mean latitude of all places north of the equator. *Ans.* 32°.7.

(11) Two points are taken at random in a triangle; find the mean area of the triangular portion which the line joining them cuts off from the whole triangle. *Ans.* $\frac{4}{9}$ of the whole.

(12) Find the mean distance of points within a sphere from a given point of the surface. *Ans.* $\frac{6}{5}a$.

(13) Find the mean distance of two points taken at random within a sphere. *Ans.* $\frac{36}{35}a$.

(14) Two points are taken at random in a given line a; find the chance that their distance shall exceed a given value c.

Ans. $\left(\dfrac{a-c}{a}\right)^2$.

(15) Find the chance that the distance of two points within a square shall not exceed a side of the square. *Ans.* $\pi - \frac{13}{6}$.

(16) A line crosses a circle at random; find the chances that a point, taken at random within the circle, shall be distant from the line by less than the radius of the circle. *Ans.* $1-\dfrac{2}{3\pi}$.

(17) A random straight line crosses a circle; find the chance that two points, taken at random in the circle, shall lie on opposite sides of the line. *Ans.* $\dfrac{128}{45\pi^2}$.

(18) A random straight line is drawn across a square; find the chance that it intersects two opposite sides. *Ans.* $\frac{1}{2}-\dfrac{\log 2}{\pi}$.

(19) Two arrows are sticking in a circular target; find the chance that their distance apart is greater than the radius.

Ans. $\dfrac{3\sqrt{3}}{4\pi}$.

(20) From a point in the circumference of a circular field a projectile is thrown at random with a given velocity which is such that the diameter of the field is equal to the greatest range of the projectile; find the chance of its falling within the field.

Ans. $\frac{1}{2}-\dfrac{2}{\pi}(\sqrt{2}-1)$.

(21) On a table a series of equidistant parallel lines is drawn, and a cube is thrown at random on the table. Supposing that the diagonal of the cube is less than the distance between consecutive straight lines, find the chance that the cube will rest without covering any part of the lines.

Ans. $1-\dfrac{4a}{\pi c}$, where a is the edge of the cube and c the distance between consecutive lines.

(22) A plane area is ruled with equidistant parallel straight lines, the distance between consecutive lines being c. A closed curve, having no singular points, whose greatest diameter is less than c, is thrown down on the area. Find the chance that the curve falls on one of the lines.

Ans. $\dfrac{l}{\pi c}$, where l is the perimeter of the curve.

(23) During a heavy rain-storm, a circular pond is formed in a circular field. If a man undertakes to cross the field in the dark, what is the chance that he will walk into the pond?

CHAPTER XV.

KEY TO THE SOLUTION OF DIFFERENTIAL EQUATIONS.

152. In this chapter an analytical key leads to a set of concise, practical rules, embodying most of the ordinary methods employed in solving differential equations; and the attempt has been made to render these rules so explicit that they may be understood and applied by any one who has mastered the Integral Calculus proper.

The key is based upon "Boole's Differential Equations" (London: Macmillan & Co.), to which the student who wishes to become familiar with the theoretical considerations upon which the working rules are based is referred.

153. A *differential equation* is an expressed relation involving derivatives with or without the primitive variables from which they are derived.

For example:

$$(1+x)y+(1-y)x\frac{dy}{dx}=0, \tag{1}$$

$$x\frac{dy}{dx}-ay=x+1, \tag{2}$$

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0, \tag{3}$$

$$D_x^2z-a^2D_y^2z=0, \tag{4}$$

are differential equations.

The *order* of a differential equation is the same as that of the derivative of highest order which appears in the equation.

Equations (1) and (2) are of the first order; (3) and (4) of the second order.

The *degree* of a differential equation is the same as the power

to which the derivative of highest order in the equation is raised, that derivative being supposed to enter into the equation in a rational form.

Equations (1), (2), (3), and (4) are all of the first degree.

A differential equation is *linear* when it would be of the first degree if the dependent variable and all its derivatives were regarded as unknown quantities.

Equations (2), (3), and (4) are *linear*.

The equation not containing differentials or derivatives, and expressing the most general relation between the primitive variables consistent with the given differential equation, is called its *general solution* or *complete primitive*. A general solution will always contain arbitrary constants or arbitrary functions.

A *singular solution* of a differential equation is a relation between the primitive variables which satisfies the differential equation by means of the values which it gives to the derivatives, but which cannot be obtained from the complete primitive by giving particular values to the arbitrary constants.

154. We shall illustrate the use of the key by solving equations (1), (2), (3), and (4) of Art. 157 by its aid.

$$(1)\quad (1+x)y+(1-y)x\frac{dy}{dx}=0, \text{ or } (1+x)ydx+(1-y)xdy=0.$$

Beginning at the beginning of the key, we see that we have a single equation, and hence look under I., p. 163; it involves ordinary derivatives: we are then directed to II., p. 163; it contains two variables: we go to III., p. 163; it is of the first order, IV., p. 163, and of the first degree, V., p. 163.

It is reducible to the form

$$\frac{1+x}{x}dx+\frac{1-y}{y}dy=0,$$

which comes under $$Xdx+Ydy=0.$$

Hence we turn to (1), p. 166, and there find the specific directions for its solution. Integrating each term separately, we get

$$\log x + x + \log y - y = c, \quad \text{or} \quad \log(xy) + x - y = c,$$

the required primitive equation.

(2) $$x\frac{dy}{dx} - ay = x + 1.$$

Beginning again at the beginning of the key, we are directed through I., II., III., IV., to V., p. 163. Looking under V., we see that it will come under either the third or the fourth head. Let us try the fourth; we are referred to (4), p. 167, for specific directions.

Obeying instructions, the work is as follows:

$$x\frac{dy}{dx} - ay = 0,$$

$$xdy - aydx = 0,$$

$$\frac{dy}{y} - \frac{adx}{x} = 0,$$

$$\log y - a\log x = c,$$

$$\log\frac{y}{x^a} = c;$$

$$\frac{y}{x^a} = C,$$

$$y = Cx^a, \tag{1}$$

$$\frac{dy}{dx} = aCx^{a-1} + x^a\frac{dC}{dx}.$$

Substitute in the given equation

$$aCx^a + x^{a+1}\frac{dC}{dx} - aCx^a = x + 1,$$

$$x^{a+1}\frac{dC}{dx} - (x + 1) = 0,$$

$$dC - \frac{x+1}{x^{a+1}}dx = 0,$$

$$C + \frac{1}{(a-1)x^{a-1}} + \frac{1}{ax^a} = C'.$$

Substitute this value for C in (1), and we get

$$y = C'x^a - \left(\frac{1}{a} + \frac{x}{a-1}\right),$$

the required primitive.

(3) $$\frac{d^2y}{dx^2} + \frac{2\,dy}{dx} = 0.$$

Beginning at the beginning of the key, we are directed through I., II., VII., to (20), p. 171, for our specific instructions.

Obeying these, our work is as follows:

$$y = Ce^{mx},$$
$$dy = mCe^{mx}\,dx,$$
$$d^2y = m^2Ce^{mx}\,dx^2.$$

Substitute in the equation, and

$$m^2Ce^{mx} + 2\,mCe^{mx} = 0,$$

or $$m^2 + 2\,m = 0;$$

$$m = 0 \quad \text{or} \quad -2.$$

$$y = C' + C''e^{-2x}$$

is the solution required.

(4) $$D_x^2z - a^2D_y^2z = 0.$$

Beginning at the beginning of the key, we are directed through I. and IX. to (43), p. 179, for our specific instructions.

Obeying these, our work is as follows:

$$dy^2 - a^2dx^2 = 0,$$
$$dy - adx = 0, \qquad (1)$$
$$dy + adx = 0, \qquad (2)$$
$$dpdy - a^2dqdx = 0. \qquad (3)$$

Combining (1) and (3), we get

$$dpdy - adqdy = 0,$$

or $$dp - adq = 0. \qquad (4)$$

(1) gives $\qquad y - ax = \alpha.$
(4) gives $\qquad p - aq = \beta.$

(2) and (3) give us, in the same way,

$$y + qx = \alpha_1,$$
$$p + aq = \beta_1;$$

and our two first integrals are

$$p - aq = f_1(y - ax), \tag{5}$$
$$p + aq = f_2(y + ax), \tag{6}$$

f_1 and f_2 denoting arbitrary functions.

Determining p and q, from (5) and (6),

$$p = \tfrac{1}{2}\ [f_2(y + ax) + f_1(y - ax)],$$
$$q = \frac{1}{2a}[f_2(y + ax) - f_1(y - ax)];$$
$$dz = \tfrac{1}{2}[f_2(y+ax) + f_1(y-ax)]\,dx + \frac{1}{2a}[f_2(y+ax) + f_1(y-ax)]\,dy$$
$$= \frac{f_2(y + ax)\,(dy + adx) - f_1(y - ax)\,(dy - adx)}{2a}.$$

Hence, $\qquad z = F(y + ax) + F_1(y - ax),$

where F and F_1 denote arbitrary functions obtained by integrating f_1 and f_2, which are arbitrary.

KEY.

* Of course, X, X_1, X_2, and Y may be constants.

Page

Of the form $\frac{dy}{dx}+X_1y=X_2y^n$, where X_1 and X_2 are functions of x alone* . . . . (5) 167

$Mdx+Ndy$ an exact differential. Test, $D_yM=D_xN$ . . . . (6) 167

$Mx+Ny=0$ . . . . (7) 168

$Mx-Ny=0$ . . . . (8) 168

Of the form $F_1(xy)ydx+F_2(xy)xdy=0$ . . (9) 168

$\frac{D_yM-D_xN}{N}$, a function of x alone . . . . (10) 168

$\frac{D_xN-D_yM}{M}$, a function of y alone . . . . (11) 168

$\frac{D_yM-D_xN}{Ny-Mx}$, a function of (xy) . . . . . (12) 168

$\frac{x^2(D_xN-D_yM)+nNx}{Mx+Ny}$, a function of $\frac{y}{x}$; n being any number . . . . (13) 168

VI. Not of first degree.

Can be solved as an algebraic equation in p, where p stands for $\frac{dy}{dx}$ . . . . (14) 169

Involves only one of the variables and p, where p stands for $\frac{dy}{dx}$ . . . . (15) 169

Of the form $xf_1p+yf_2p=f_3p$, where p stands for $\frac{dy}{dx}$ . . . . (16) 169

Homogeneous relatively to x and y . . . . (17) 170

Of the form $F(\phi,\psi)=0$, where ϕ and ψ are functions of x, y, and $\frac{dy}{dx}$, such that $\phi=a$ $\psi=b$ will lead, on differentiation, to the same differential equation of the second order . . . . (18) 170

A singular solution will answer . . . . . (19) 170

* See note, p. 163.

* The first member is supposed to contain only those terms involving the dependent variable or its derivatives.

† See note, p. 163.

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(1) Of or reducible to the form $Xdx + Ydy = 0$, where X is a function of x alone and Y is a function of y alone.

Integrate each term separately, and write the sum of their integrals equal to an arbitrary constant.

(2) M and N homogeneous functions of x and y of the same degree.

Introduce in place of y the new variable v defined by the equation $y = vx$, and the equation thus obtained can be solved by (1).

Or, multiply the equation through by $\dfrac{1}{Mx + Ny}$, and its first member will become an exact differential, and the solution may be obtained by (6).

(3) Of the form $(ax + by + c)dx + (a'x + b'y + c')dy = 0$.

If $ab' - a'b = 0$, the equation may be thrown into the form $(ax + by + c)dx + \frac{a'}{a}(ax + by + c)dy = 0$. If now $z = ax + by$ be introduced in place of either x or y, the resulting equation can be solved by (1).

If $ab' - a'b$ does not equal zero, the equation can be made homogeneous by assuming $x = x' - \alpha$, $y = y' - \beta$, and determining α and β so that the constant terms in the new values of M and N shall disappear, and it can then be solved by (2).

(4) Linear. General form $\frac{dy}{dx} + X_1 y = X_2$, where X_1 and X_2 are functions of x alone.

Solve on the supposition that $X_2 = 0$ by (1); and from this solution obtain a value for y involving of course an arbitrary constant C. Substitute this value of y in the given equation, regarding C as a variable, and there will result a differential equation, involving C and x, whose solution by (1) will express C as a function of x. Substitute this value for C in the expression already obtained for y, and the result will be the required solution.

(5) Of the form $\frac{dy}{dx} + X_1 y = X_2 y^n$, where X_1 and X_2 are functions of x alone.

Divide through by y^n, and then introduce $z = y^{1-n}$ in place of y, and the equation will become linear and may be solved by (4).

(6) $Mdx + Ndy$ an exact differential. Test $D_y M = D_x N$.

Find $\int Mdx$, regarding y as constant, and add an arbitrary function of y. Determine this function of y by the fact that the differential of the result just mentioned, taken on the supposition that x is constant, must equal Ndy. Write equal to an arbitrary constant the $\int Mdx$ above mentioned plus the function of y just determined.

(7) $Mx + Ny = 0$.

Multiply the equation through by $\dfrac{1}{Mx - Ny}$, and the first member will become an exact differential. The solution may then be found by (6).

(8) $Mx - Ny = 0$.

Multiply the equation through by $\dfrac{1}{Mx + Ny}$, and the first member will become an exact differential. The solution may then be found by (6).

(9) Of the form $f_1(xy)ydx + f_2(xy)xdy = 0$.

Multiply through by $\dfrac{1}{Mx - Ny}$, and the first member will become an exact differential. The solution may then be found by (6).

(10) $\dfrac{D_yM - D_xN}{N}$, a function of x alone.

Multiply the equation through by $e^{\int\frac{D_yM - D_xN}{N}\cdot dx}$, and the first member will become an exact differential. The solution may then be found by (6).

(11) $\dfrac{D_xN - D_yM}{M}$, a function of y alone.

Multiply the equation through by $e^{\int\frac{D_xN - D_yM}{M}\cdot dy}$, and the first member will become an exact differential. The solution may then be found by (6).

(12) $\dfrac{D_yM - D_xN}{Ny - Mx}$, a function of (xy).

Multiply the equation through by $e^{\int\frac{D_yM - D_xN}{Ny - Mx}\cdot dv}$ where $v = xy$, and the first member will become an exact differential. The solution may thus be found by (6).

(13) $\dfrac{x^2(D_xN - D_yM) + nNx}{Mx + Ny}$, a function of $\dfrac{y}{x}$; n being any number.

Multiply the equation through by $x^n e^{\int fv.dv}$, where $v = \frac{y}{x}$, and $fv = \frac{x^2(D_x N - D_y M) + nNx}{Mx + Ny}$, and the first member will become an exact differential. The solution may then be found by (6).

(14) Can be solved as an algebraic equation in p, where p stands for $\frac{dy}{dx}$.

Solve as an algebraic equation in p, and, after transposing all the terms to the first member, express the first member as the product of factors of the first order and degree. Write each of these factors separately equal to zero, and find its solution in the form $V - c = 0$ by (V.). Write the product of the first members of these solutions equal to zero, using the same arbitrary constant in each.

(15) Involves only one of the variables and p, where p stands for $\frac{dy}{dx}$.

By algebraic solution express the variable as an explicit function of p, and then differentiate through relatively to the other variable, regarding p as a new variable and remembering that $\frac{dx}{dy} = \frac{1}{p}$. There will result a differential equation of the first order and degree between the second variable and p which can be solved by (1). Eliminate p between this solution and the given equation, and the resulting equation will be the required solution.

(16) Of the form $xf_1p + yf_2p = f_3p$, where p stands for $\frac{dy}{dx}$.

Differentiate the equation relatively to one of the variables, regarding p as a new variable, and, with the aid of the given equation, eliminate the other original variable. There will result a linear differential equation of the first order between p and the remaining variable, which may be simplified by striking out any factor not containing $\frac{dp}{dx}$ or

$\frac{dp}{dy}$, and can be solved by (4). Eliminate p between this solution and the given equation, and the result will be the required solution.

(17) Homogeneous relatively to x and y.

Let $y = vx$, and solve algebraically relatively to p or v, p standing for $\frac{dy}{dx}$. The result will be of the form $p = fv$ or $v = Fp$. If $p = fv$, $\frac{dy}{dx} = fv$, $\frac{d(vx)}{dx} = fv$, $x\frac{dv}{dx} + v = fv$, an equation that can be solved by (1). If $v = Fp$, $\frac{y}{x} = Fp$, $y = xFp$, an equation that can be solved by (16).

(18) Of the form $F(\phi,\psi) = 0$, where ϕ and ψ are functions of x, y and $\frac{dy}{dx}$, such that $\phi = a$ and $\psi = b$ will lead, on differentiation, to the same differential equations of the second order.

Eliminate $\frac{dy}{dx}$ between $\phi = a$ and $\psi = b$, where a and b are arbitrary constants subject to the relation that $F(a,b) = 0$, and the result will be the required solution.

(19) Singular solution will answer.

Let $\frac{dy}{dx} = p$ and express p as an explicit function of x and y. Take $\frac{dp}{dy}$, regarding x as constant, and see whether it can be made infinite by writing equal to zero any expression involving y. If so, and if the equation thus formed will satisfy the given differential equation, it is a singular solution.

Or take $\frac{d\left(\frac{1}{p}\right)}{dx}$, regarding y as constant, and see whether it can be made infinite by writing equal to zero any expression involving x. If so, and if the equation thus formed is consistent with the given equation, it is a singular solution.

(20) Linear, with constant coefficients. Second member zero.

Assume $y = Ce^{mx}$; C and m being constants, substitute in the given equation, and then divide through by Ce^{mx}. There will result an algebraic equation in m. Solve this equation, and the complete value of y will consist of a series of terms characterized as follows: For every distinct real value of m there will be a term Ce^{mx}; for each pair of imaginary values, $a + b\sqrt{-1}$, $a - b\sqrt{-1}$, a term $Ae^{ax}\cos bx + Be^{ax}\sin bx$; each of the coefficients A, B, and C being an arbitrary constant, if the root or pair of roots occurs but once, and an algebraic polynomial in x of the $(r-1)$st degree with arbitrary constant coefficients, if the root or pair of roots occurs r times.

(21) Linear, with constant coefficients. Second member not zero.

Solve, on the hypothesis that the second member is zero, and obtain the complete value of y by (20). Denoting the order of the given equation by n, form the $n-1$ successive derivatives $\frac{dy}{dx}, \frac{d^2y}{dx^2} \ldots\ldots \frac{d^{n-1}y}{dx^{n-1}}$. Then differentiate y and each of the values just obtained, regarding the arbitrary constants as new variables, and substitute the resulting values in the given equation; and by its aid, and the $n-1$ equations of condition formed by writing each of the derivatives of the second set, except the nth, equal to the derivative of the same order in the first set, determine the arbitrary coefficients and substitute their values in the original expression for y.

Or, if the second member of the given equation can be got rid of by differentiation, or by differentiation and elimination, between the given and the derived equations, solve the new differential equation thus obtained by (20), and determine the superfluous arbitrary constants so that the given equation shall be satisfied.

(22) Of the form $(a+bx)^n\frac{d^ny}{dx^n}+A(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\cdots\cdots+Ly=X$, where X is a function of x alone.

Assume $a+bx=e^t$, and change the independent variable in the given equation so as to introduce t in place of x. The solution can then be obtained by (21).

(23) Either of the primitive variables wanting.

Assume z equal to the derivative of lowest order in the equation, and express the equation in terms of z and its derivatives with respect to the primitive variable actually present, and the order of the resulting equation will be lower than that of the given one.

(24) Of the form $\frac{d^ny}{dx^n}=X$. X being a function of x alone.

Solve by integrating n times successively with regard to x.

Or solve by (21).

(25) Of the form $\frac{d^2y}{dx^2}=Y$. Y being a function of y alone.

Multiply by $2\frac{dy}{dx}$ and integrate relatively to x. There will result the equation $\left(\frac{dy}{dx}\right)^2=2\int Ydy+C$, whence $\frac{dy}{dx}=(2\int Ydy+C)^{\frac{1}{2}}$, an equation that may be solved by (1).

(26) Of the form $\frac{d^ny}{dx^n}=f\frac{d^{n-1}y}{dx^{n-1}}$.

Assume $\frac{d^{n-1}y}{dx^{n-1}}=z$, then $\frac{dz}{dx}=fz$ or $dx=\frac{dz}{fz}$, $x=\int\frac{dz}{fz}+C$. After effecting this integration, express z in terms of x and C. Then, since $z=\frac{d^{n-1}y}{dx^{n-1}}$, $\frac{d^{n-1}y}{dx^{n-1}}=F(x,C)$, an equation that may be treated by (24).

Or, since $\frac{d^{n-1}y}{dx^{n-1}}=z$, $\frac{d^{n-2}y}{dx^{n-2}}=\int zdx+c=\int\frac{zdz}{fz}+c$, since

$dx = \frac{dz}{fz}$. $\frac{d^{n-3}y}{dx^{n-3}} = \int dx\left(\int \frac{zdz}{fz} + c\right) + c_1 = \int \frac{dz}{fz}\left(\int \frac{zdz}{fz} + c\right)$ $+ c_1$, ... Continue this process until y is expressed in terms of z and $n-1$, arbitrary constants, and then eliminate z by the aid of the equation $x = \int \frac{dz}{fz} + C$.

(27) Of the form $\frac{d^n y}{dx^n} = f\frac{d^{n-2}y}{dx^{n-2}}$.

Let $\frac{d^{n-2}y}{dx^{n-2}} = z$, and the equation becomes $\frac{d^2 z}{dx^2} = fz$, and may be solved by (25).

(28) Homogeneous on the supposition that x and y are of the degree 1, $\frac{dy}{dx}$ of the degree 0, $\frac{d^2 y}{dx^2}$ of the degree -1, ...

Assume $x = e^\theta$, $y = e^\theta z$, and by changing the variables introduce θ and z into the equation in the place of x and y. Divide through by e^θ and there will result an equation involving only z, $\frac{dz}{d\theta}$, $\frac{d^2 z}{d\theta^2}$, ..., whose order may be depressed by (23).

(29) Homogeneous on the supposition that x is of the degree 1, y of the degree n, $\frac{dy}{dx}$ of the degree $n-1$, $\frac{d^2 y}{dx^2}$ of the degree $n-2$, ...

Assume $x = e^\theta$, $y = e^{n\theta} z$, and by changing the variables introduce θ and z into the equation in the place of x and y. The resulting equation may be freed from θ by division and treated by (23).

(30) Homogeneous relatively to y, $\frac{dy}{dx}$, $\frac{d^2 y}{dx^2}$, ...

Assume $y = e^z$, and substitute in the given equation. Divide through by e^z and treat by (23).

(31) Containing the first power only of the derivative of the highest order.

The first member of the equation may be an exact derivative; call it $\frac{dV}{dx}$. If n is the order of the equation,

represent $\frac{d^{n-1}y}{dx^{n-1}}$ by p and $\frac{d^n y}{dx^n}$ by $\frac{dp}{dx}$. Multiply the term containing $\frac{dp}{dx}$ by dx and integrate it as if p were the only variable, calling the result U_1; then replacing p by $\frac{d^{n-1}y}{dx^{n-1}}$, find the complete derivative $\frac{dU_1}{dx}$, and form the expression $\frac{dV}{dx}-\frac{dU_1}{dx}$, representing it by $\frac{dV_1}{dx}$. If $\frac{dV_1}{dx}$ contains the first power only of the highest derivative of y, it may itself be an exact derivative, and is to be treated precisely as the first member of the given equation $\frac{dV}{dx}$ has been. Continue this process until a remainder $\frac{dV_{n-1}}{dx}$ of the first order occurs. Write this equal to zero, and solve by (6), throwing its solution into the form $V_{n-1}=C$. A complete first integral of the given equation will be $U_1+U_2+\cdots\cdots+V_{n-1}=C$. The occurrence at any step of the process of a remainder $\frac{dV_k}{dx}$, containing a higher power than the first of its highest derivative of y, shows that the first member of the given equation was not an exact derivative, and that this method will not apply.

(32) Of the form $\frac{d^2y}{dx^2}+X\frac{dy}{dx}+Y\left[\frac{dy}{dx}\right]^2=0$, where X is a function of x alone and Y a function of y alone. Multiply through by $\left[\frac{dy}{dx}\right]^{-1}$ and the equation will become exact, and may be solved by (31).

(33) Singular integral will answer.

Call $\frac{d^{n-1}y}{dx^{n-1}}$ p, and $\frac{d^n y}{dx^n}$ q, and find $\frac{dq}{dp}$, regarding p and q as the only variables, and see whether $\frac{dq}{dp}$ can be made infinite by writing equal to zero any factor containing p.

If so, eliminate q between this equation and the given equation, and if the result is a solution it will be a singular integral.

(34) General form, $Pdx + Qdy + Rdz = 0$.

If the equation can be reduced to the form $Xdx + Ydy + Zdz = 0$, where X is a function of x alone, Y a function of y alone, and Z a function of z alone, integrate each term separately, and write the sum of the integrals equal to an arbitrary constant.

If not, integrate the equation by (V.) on the supposition that one of the variables is constant and its differential zero, writing an arbitrary function of that variable in place of the arbitrary constant in the result. Transpose all the terms to the first member, and then take its complete differential, regarding all the original variables as variable, and write it equal to the first member of the given equation, and from this equation of condition determine the arbitrary function. Substitute for the arbitrary function in the first integral its value thus determined, and the result will be the solution required.

If the equation of condition contains any other variables than the one involved in the arbitrary function, they must be eliminated by the aid of the primitive equation already obtained; and if this elimination cannot be performed, the given equation is not derivable from a single primitive equation, but must have come from two simultaneous primitive equations.

In that case, assume any arbitrary equation $f(x,y,z) = 0$ as one primitive, differentiate it, and eliminate between it its derived equation and the given equation, one variable, and its differential. There will result a differential equation containing only two variables, which may be solved by (III.), and will lead to the second primitive of the given equation.

(35) General form, $Pdx_1 + Qdx_2 + Rdx_3 + \cdots\cdots = 0$.

If the equation can be reduced to the form $X_1dx_1 + X_2dx_2 + X_3dx_3 + \cdots\cdots = 0$, where X_1 is a function of x_1 alone, X_2 a function of x_2 alone, X_3 a function of x_3 alone, etc., integrate each term separately, and write the sum of their integrals equal to an arbitrary constant.

If not, integrate the equation by (V.), on the supposition that all the variables but two are constant and their differentials zero, writing an arbitrary function of these variables in place of the arbitrary constant in the result. Transpose all the terms to the first member, and then take its complete differential, regarding all of the original variables as variable, and write it equal to the first member of the given equation, and from this equation of condition determine the arbitrary function. Substitute for the arbitrary function in the first integral its value thus determined, and the result will be the solution required.

If the equation of condition cannot, even with the aid of the primitive equation first obtained, be thrown into a form where the complete differential of the arbitrary function is given equal to an exact differential, the function cannot be determined, and the given equation is not derivable from a single primitive equation.

(36) System of simultaneous equations of the first order.

If any of the equations of the set can be integrated separately by (II.) so as to lead to single primitives, the problem can be simplified; for by the aid of these primitives a number of variables equal to the number of solved equations can be eliminated from the remaining equations of the series, and there will be formed a simpler set of simultaneous equations whose primitives, together with the primitives already found, will form the primitive system of the given equations.

There must be n equations connecting $n+1$ variables, in order that the system may be determinate.

Let x, x_1, x_2, ..., x_n be the original variables. Choose

any two, x and x_1, as the independent and the principal dependent variable, and by successive eliminations form the n equations $\frac{dx_1}{dx}=f_1(x,x_1,x_2,\ldots\ldots,x_n)$, $\frac{dx_2}{dx}=f_2(x,x_1,x_2,\ldots\ldots,x_n)$, ..., up to $\frac{dx_n}{dx}=f_n(x,x_1,x_2,\ldots\ldots,x_n)$. Differentiate the first of these with respect to x $n-1$ times, substituting for $\frac{dx_2}{dx}, \frac{dx_3}{dx}, \ldots\ldots, \frac{dx_n}{dx}$, after each step their values in terms of the original variables. There will result n equations, which will express each of the n successive derivatives $\frac{dx_1}{dx}, \frac{d^2x_1}{dx^2}, \frac{d^3x_1}{dx^3}, \ldots\ldots, \frac{d^nx_1}{dx^n}$, in terms of x, x_1, x_2, ..., x_n. Eliminate from these all the variables except x and x_1, obtaining a single equation of the nth order between x and x_1. Solve this by (VII.), and so get a value of x_1 in terms of x and n arbitrary constants. Find by differentiating this result values for $\frac{dx_1}{dx}, \frac{d^2x_1}{dx^2}, \ldots\ldots, \frac{d^{n-1}x_1}{dx^{n-1}}$, and write them equal to the ones already obtained for them in terms of the original variables. The $n-1$ equations thus formed, together with the equation expressing x_1 in terms of x and arbitrary constants, are the complete primitive system required.

(37) System of simultaneous equations not of the first order. Regard each derivative of each dependent variable, from the first to the next to the highest as a new variable, and the given equations, together with the equations defining these new variables, will form a system of simultaneous equations of the first order which may be solved by (36). Eliminate the new variables representing the various derivatives from the equations of the solution, and the equations obtained will be the complete primitive system required.

(38) All the partial derivatives taken with respect to one of the independent variables.

Integrate by (II.) as if that one were the only independent variable, replacing each arbitrary constant by an arbitrary function of the other independent variables.

(39) Of the first order and linear, containing three variables. General form, $PD_xz + QD_yz = R$.

Form the auxiliary system of ordinary differential equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$, and integrate by (36). Express their primitives in the form $u = a$, $v = b$, a and b being arbitrary constants; and $u = fv$, where f is an arbitrary function, will be the required solution.

(40) Of the first order and linear, containing more than three variables. General form, $P_1 D_{x_1} z + P_2 D_{x_2} z + \cdots\cdots = R$, where x_1, x_2, $\cdots\cdots$, x_n, are the independent and z the dependent variables.

Form the auxiliary system of ordinary differential equations $\frac{dx_1}{P_1} = \frac{dx_2}{P_2} \cdots\cdots = \frac{dx_n}{P_n} = \frac{dz}{R}$, and integrate them by (36). Express their primitives in the form $v_1 = a$, $v_2 = b$, $v_3 = c$, $\cdots\cdots$, and $v_1 = f(v_2, v_3, \cdots\cdots, v_n)$, where f is an arbitrary function, will be the required solution.

(41) Of the first order and not linear, containing three variables, $F(x,y,z,p,q) = 0$, where $p = D_xz$, $q = D_yz$.

Express q in terms of x, y, z and p from the given equation, and substitute its value thus obtained in the auxiliary system of ordinary differential equations $\frac{dx}{-D_pq} = dy = \frac{dz}{q - pD_pq} = \frac{dp}{D_xq + pD_zq}$. Deduce by integration from these equations, by (36), a value of p involving an arbitrary constant, and substitute it with the corresponding value of q in the equation $dz = pdx + qdy$. Integrate this result by (34), if possible; and if a single primitive equation be obtained, it will be a complete primitive of the given equation.

A singular solution may be obtained by finding the partial derivatives $D_p z$ and $D_q z$ from the given equation, writing them separately equal to zero, and eliminating p and q between them and the given equation.

(42) Of the first order and not linear, containing more than three variables. $F(x_1, x_2, \cdots\cdots, x_n, z, p_1, p_2, \cdots\cdots, p_n) = 0$, where $p_1 = D_{x_1}z$, $p_2 = D_{x_2}z$, $\cdots\cdots$.

Form the linear partial differential equation $\Sigma_i[(D_{x_i}F + p_i D_z F)D_{p_i}\Phi - D_{p_i}F(D_{x_i}\Phi + p_i D_z\Phi)] = 0$, where Φ is an unknown function of $(x_1, \cdots\cdots, x_n, p_1, \cdots\cdots, p_n)$, and where Σ_i means the sum of all the terms of the given form that can be obtained by giving i successively the values 1, 2, 3, $\cdots\cdots$, n.

Form, by (40), its auxiliary system of ordinary differential equations, and from them get, by (36), $n-1$ integrals, $\Phi_1 = a_1$, $\Phi_2 = a_2, \cdots\cdots, \Phi_{n-1} = a_{n-1}$. By these equations and the given equation express p_1, p_2, $\cdots\cdots$, p_n in terms of the original variables, and substitute their values in the equation $dz = p_1 dx_1 + p_2 dx_2 + \cdots\cdots + p_n dx_n$. Integrate this by (35), and the result will be the required complete primitive.

(43) Of the second order and containing the derivatives of the second order only in the first degree. General form, $RD_x^2 z + SD_x D_y z + TD_y^2 z = V$, where R, S, T, and V may be functions of x, y, z, $D_x z$, and $D_y z$.

Call $D_x z$ p and $D_y z$ q.

Form first the equation

$$Rdy^2 - Sdxdy + Tdx^2 = 0, \qquad [1]$$

and resolve it, supposing the first member not a complete square, into two equations of the form

$$dy - m_1 dx = 0, \qquad dy - m_2 dx = 0. \qquad [2]$$

From the first of these, and from the equation

$$Rdpdy + Tdqdx - Vdxdy = 0, \qquad [3]$$

combined if needful with the equation

$$dz = pdx + qdy,$$

seek to obtain two integrals $u_1 = \alpha$, $v_1 = \beta$. Proceeding in the same way with the second equation of [2], seek two other integrals $u_2 = \alpha_1$, $v_2 = \beta_1$; then the first integrals of the proposed equation will be

$$u_1 = f_1 v_1, \qquad u_2 = f_2 v_2, \tag{4}$$

where f_1 and f_2 denote arbitrary functions.

To deduce the final integral, we must either integrate one of these, or, determining from the two p and q in terms of x, y, and z, substitute those values in the equation

$$dz = pdx + qdy,$$

which will then become integrable. Its solution will give the final integral sought.

If the values of m_1 and m_2 are equal, only one first integral will be obtained, and the final solution must be sought by its integration.

When it is not possible so to combine the auxiliary equations as to obtain two auxiliary integrals $u = \alpha$, $v = \beta$, no first integral of the proposed equation exists, and this method of solution fails.

Examples.

(1) $\sin x \cos y dx - \cos x \sin y dy = 0$. *Ans.* $\cos y = c \cos x$.

(2) $[2\sqrt{(xy)} - x]\,dy + ydx = 0$. *Ans.* $y = ce^{-\sqrt{\frac{x}{y}}}$.

(3) $(2x - y + 1)dx + (2y - x - 1)dy = 0$.
Ans. $x^2 - xy + y^2 + x - y = c$.

(4) $\dfrac{dy}{dx} + y\cos x = \dfrac{\sin 2x}{2}$. *Ans.* $y = \sin x - 1 + ce^{-\sin x}$.

(5) $(1-x^2)\dfrac{dy}{dx}-xy=axy^2.$ *Ans.* $y=[c\sqrt{(1-x^2)}-a]^{-1}.$

(6) $xdx+ydy+\dfrac{xdy-ydx}{x^2+y^2}=0.$ *Ans.* $\dfrac{x^2+y^2}{2}+\tan^{-1}\dfrac{y}{x}=c.$

(7) $y=x\dfrac{dy}{dx}+\dfrac{dy}{dx}-\left(\dfrac{dy}{dx}\right)^2.$ *Ans.* $y=cx+c-c^2.$

Singular solution, $y=\dfrac{(x+1)^2}{4}.$

(8) $\left(\dfrac{dy}{dx}\right)^2-\dfrac{a^2}{x^2}=0.$ *Ans.* $(y-a\log x-c)(y+a\log x-c)=0.$

(9) $\dfrac{d^4y}{dx^4}-4\dfrac{d^3y}{dx^3}+6\dfrac{d^2y}{dx^2}-4\dfrac{dy}{dx}+y=0.$

Ans. $y=(c_0+c_1x+c_2x^2+c_3x^3)e^x.$

(10) $\dfrac{d^2y}{dx^2}-2k\dfrac{dy}{dx}+k^2y=e^x.$ *Ans.* $y=(c_1+c_2x)e^{kx}+\dfrac{e^x}{(k-1)^2}.$

(11) $\dfrac{d^2y}{dx^2}+\dfrac{1}{x}\dfrac{dy}{dx}=0.$ *Ans.* $y=c\log x+c'.$

(12) $x^2\dfrac{d^3y}{dx^3}+x\dfrac{d^2y}{dx^2}+(2xy-1)\dfrac{dy}{dx}+y^2=0.$ Find a first integral.

Ans. $x^2\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}+xy^2=0.$

(13) $\dfrac{dx}{x-a}+\dfrac{dy}{y-b}+\dfrac{dz}{z-c}=0.$ *Ans.* $(x-a)(y-b)(z-c)=C.$

(14) $(y+z)dx+dy+dz=0.$ *Ans.* $e^x(y+z)=c.$

(15) $\dfrac{dx}{dt}+4x+\dfrac{y}{4}=0,\quad \dfrac{dy}{dt}+3y-x=0.$

Ans. $x=ce^{-\frac{7t}{2}}-\dfrac{y}{2},\quad y=(ct+c_1)\,e^{-\frac{7t}{2}}.$

(16) $\dfrac{d^2x}{dt^2}+m^2x=0,\quad \dfrac{d^2y}{dt^2}-m^2x=0.$ *Ans.*

(17) $D_xz=\dfrac{y}{x+z}.$ *Ans.* $e^{-\frac{z}{y}}(x+y+z)=\phi y.$

(18) $xzD_xz + yzD_yz = xy$. *Ans.* $z^2 = xy + \phi\left(\frac{x}{y}\right)$.

(19) $D_xz.D_yz = 1$. *Ans.* $z = ax + \frac{y}{a} + b$.

(20) $x^2D_x^2z + 2xyD_xD_yz + y^2D_y^2z = 0$. *Ans.* $z = x\phi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right)$.

(21) $(D_yz)^2D_x^2z - 2D_xzD_yzD_xD_yz + (D_xz)^2D_y^2z = 0$.
Ans. $y = x\phi z + \psi z$.

(22) $D_xz.D_xD_yz - D_yz.D_x^2z = 0$. *Ans.* $x = \phi y + \psi z$.

APPENDIX.

CHAPTER V.

INTEGRATION.

74. We are now able to extend materially our list of *formulas for direct integration* (Art. 55), one of which may be obtained from each of the derivative formulas in our last chapter. The following set contains the most important of these: —

$D_x \log x = \frac{1}{x}$ gives $\int_x \frac{1}{x} = \log x.$

$D_x a^x = a^x \log a$ " $\int_x a^x \log a = a^x.$

$D_x e^x = e^x$ " $\int_x e^x = e^x.$

$D_x \sin x = \cos x$ " $\int_x \cos x = \sin x.$

$D_x \cos x = -\sin x$ " $\int_x (-\sin x) = \cos x.$

$D_x \log \sin x = \operatorname{ctn} x$ " $\int_x \operatorname{ctn} x = \log \sin x.$

$D_x \log \cos x = -\tan x$ " $\int_x (-\tan x) = \log \cos x.$

$D_x \sin^{-1} x = \frac{1}{\sqrt{(1-x^2)}}$ " $\int_x \frac{1}{\sqrt{(1-x^2)}} = \sin^{-1} x.$

$D_x \tan^{-1} x = \frac{1}{1+x^2}$ " $\int_x \frac{1}{1+x^2} = \tan^{-1} x.$

$D_x \operatorname{vers}^{-1} x = \frac{1}{\sqrt{(2x-x^2)}}$ " $\int_x \frac{1}{\sqrt{(2x-x^2)}} = \operatorname{vers}^{-1} x.$

The second, fifth, and seventh in the second group can be written in the more convenient forms,

$$\int_x a^x = \frac{a^x}{\log a};$$

$$\int_x \sin x = -\cos x;$$

$$\int_x \tan x = -\log\cos x.$$

75. When the expression to be integrated does not come under any of the forms in the preceding list, *it can often be prepared for integration by a suitable change of variable*, the new variable, of course, being a function of the old. This method is called *integration by substitution*, and is based upon a formula easily deduced from $D_x(Fy) = D_y Fy \,.\, D_x y$;

which gives immediately

$$Fy = \int_x (D_y Fy . D_x y).$$

Let $u = D_y Fy,$

then $Fy = \int_y u,$

and we have $\int_y u = \int_x (u D_x y)$;

or, interchanging x and y,

$$\int_x u = \int_y (u D_y x). \qquad [1]$$

For example, required $\int_x (a+bx)^n$.

Let $z = a + bx,$

and then $\int_x (a+bx)^n = \int_x z^n = \int_z (z^n \,.\, D_z x),$ by [1];

but $x = \frac{z}{b} - \frac{a}{b},$

$$D_z x = \frac{1}{b};$$

hence $\int_x (a+bx)^n = \frac{1}{b}\int_z z^n = \frac{1}{b}\,\frac{z^{n+1}}{n+1}.$

Substituting for z its value, we have

$$\int_x (a+bx)^n = \frac{1}{b}\,\frac{(a+bx)^{n+1}}{n+1}.$$

EXAMPLE.

Find $\int_x \frac{1}{a+bx}$. *Ans.* $\frac{1}{b}\log(a+bx)$.

76. If fx represents a function that can be integrated, $f(a+bx)$ can always be integrated; for, if

$$z = a + bx,$$

then
$$D_z x = \frac{1}{b}$$

and
$$\int_x f(a+bx) = \int_x fz = \int_z fz D_z x = \frac{1}{b}\int_z fz.$$

EXAMPLES.

Find

(1) $\int_x \sin ax$. *Ans.* $-\frac{1}{a}\cos ax$.

(2) $\int_x \cos ax$. *Ans.* $\frac{1}{a}\sin ax$.

(3) $\int_x \tan ax$.

(4) $\int_x \operatorname{ctn} ax$.

77. *Required* $\int_x \frac{1}{\sqrt{(a^2-x^2)}}$.

$$\int_x \frac{1}{\sqrt{(a^2-x^2)}} = \frac{1}{a}\int_x \frac{1}{\sqrt{\left[1-\left(\frac{x}{a}\right)^2\right]}}.$$

Let
$$z = \frac{x}{a},$$

then
$$x = az,$$

$$D_z x = a,$$

$$\frac{1}{a}\int_x \frac{1}{\sqrt{\left[1-\left(\frac{x}{a}\right)^2\right]}} = \frac{1}{a}\int_x \frac{1}{\sqrt{(1-z^2)}} = \frac{1}{a}\int_z \frac{1}{\sqrt{(1-z^2)}}\, D_z x$$

$$= \int_z \frac{1}{\sqrt{(1-z^2)}} = \sin^{-1} z = \sin^{-1}\frac{x}{a}.$$

EXAMPLES.

Find

(1) $\int_x \frac{1}{a^2+x^2}$. *Ans.* $\frac{1}{a}\tan^{-1}\frac{x}{a}$.

(2) $\int_x \frac{1}{\sqrt{(2ax-x^2)}}$. *Ans.* $\operatorname{vers}^{-1}\frac{x}{a}$.

78. *Required* $\int_x \frac{1}{\sqrt{(x^2+a^2)}}$.

Let $$z = x + \sqrt{(x^2+a^2)};$$

then $$z - x = \sqrt{(x^2+a^2)},$$

$$z^2 - 2zx + x^2 = x^2 + a^2,$$

$$2zx = z^2 - a^2,$$

$$x = \frac{z^2-a^2}{2z},$$

$$\sqrt{(x^2+a^2)} = z - x = z - \frac{z^2-a^2}{2z} = \frac{z^2+a^2}{2z},$$

$$D_z x = \frac{z^2+a^2}{2z^2}.$$

$$\int_x \frac{1}{\sqrt{(x^2+a^2)}} = \int_x \frac{2z}{z^2+a^2} = \int_z \frac{2z}{z^2+a^2}\, D_z x$$

$$= \int_z \frac{2z}{z^2+a^2}\,\frac{z^2+a^2}{2z^2} = \int_z \frac{1}{z} = \log z = \log(x + \sqrt{x^2+a^2}).$$

EXAMPLE.

Find $\int_x \frac{1}{\sqrt{(x^2-a^2)}}$. *Ans.* $\log(x+\sqrt{x^2-a^2})$.

79. *When the expression to be integrated can be factored*, the required integral can often be obtained by the use of a formula

deduced from $$D_x(uv) = uD_xv + vD_xu,$$

which gives $$uv = \int_x uD_xv + \int_x vD_xu$$

or $$\int_x uD_xv = uv - \int_x vD_xu. \qquad [1]$$

This method is called *integrating by parts*.

(*a*) For example, required $\int_x \log x$.

$\log x$ can be regarded as the product of $\log x$ by 1.

Call $$\log x = u \text{ and } 1 = D_xv,$$

then $$D_xu = \frac{1}{x},$$

$$v = x;$$

and we have

$$\int_x \log x = \int_x 1 \log x = \int_x uD_xv = uv - \int_x vD_xu$$

$$= x\log x - \int_x \frac{x}{x} = x\log x - x.$$

EXAMPLE.

Find $\int_x x\log x$.

Suggestion: Let $\log x = u$ and $x = D_xv$.

Ans. $\frac{1}{2}x^2\left(\log x - \frac{1}{2}\right)$.

80. *Required* $\int_x \sin^2 x$.

Let $$u = \sin x \text{ and } D_xv = \sin x,$$

then $$D_xu = \cos x,$$

$$v = -\cos x,$$

$$\int_x \sin^2 x = -\sin x\cos x + \int_x \cos^2 x;$$

but $$\cos^2 x = 1 - \sin^2 x,$$

so $$\int_x \cos^2 x = \int_x 1 - \int_x \sin^2 x = x - \int_x \sin^2 x$$

and $$\int_x \sin^2 x = x - \sin x \cos x - \int_x \sin^2 x.$$

$$2\int_x \sin^2 x = x - \sin x \cos x.$$

$$\int_x \sin^2 x = \tfrac{1}{2}(x - \sin x \cos x).$$

EXAMPLES.

(1) Find $\int_x \cos^2 x$. *Ans.* $\dfrac{1}{2}(x + \sin x \cos x)$.

(2) $\int_x \sin x \cos x$. *Ans.* $\dfrac{\sin^2 x}{2}$.

81. *Very often both methods described above are required in the same integration.*

(*a*) *Required* $\int_x \sin^{-1} x$.

Let $$\sin^{-1} x = y,$$

then $$x = \sin y;$$

$$D_y x = \cos y,$$

$$\int_x \sin^{-1} x = \int_x y = \int_y y \cos y.$$

Let $$u = y \text{ and } D_y v = \cos y;$$

then $$D_y u = 1,$$

$$v = \sin y,$$

and
$$\int_y y \cos y = y \sin y - \int_y \sin y = y \sin y + \cos y = x \sin^{-1} x + \sqrt{(1 - x^2)}.$$

Any inverse or anti-function can be integrated by this method if the direct function is integrable.

(*b*) Thus, $$\int_x f^{-1} x = \int_x y = \int_y y D_y f y = y f y - \int_y f y$$

where $$y = f^{-1} x.$$

EXAMPLES.

(1) Find $\int_x \cos^{-1}x$. *Ans.* $x\cos^{-1}x - \sqrt{(1-x^2)}$.

(2) $\int_x \tan^{-1}x$. *Ans.* $x\tan^{-1}x - \frac{1}{2}\log(1+x^2)$.

(3) $\int_x \text{vers}^{-1}x$. *Ans.* $(x-1)\,\text{vers}^{-1}x + \sqrt{(2x-x^2)}$.

82. Sometimes an *algebraic transformation, either alone or in combination with the preceding methods, is useful.*

(*a*) *Required* $\int_x \frac{1}{x^2-a^2}$.

$$\frac{1}{x^2-a^2} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right),$$

and, by Art. 75 (Ex.),

$$\int_x \frac{1}{x^2-a^2} = \frac{1}{2a}\left[\log(x-a) - \log(x+a)\right] = \frac{1}{2a}\log\frac{x-a}{x+a}.$$

(*b*) *Required* $\int_x \sqrt{\left(\frac{1+x}{1-x}\right)}$.

$$\sqrt{\left(\frac{1+x}{1-x}\right)} = \frac{1+x}{\sqrt{(1-x^2)}} = \frac{1}{\sqrt{(1-x^2)}} + \frac{x}{\sqrt{(1-x^2)}},$$

$$\int_x \frac{1}{\sqrt{(1-x^2)}} = \sin^{-1}x.$$

$\int_x \frac{x}{\sqrt{(1-x^2)}}$ can be readily obtained by *substituting* $y=(1-x^2)$, and is $-\sqrt{(1-x^2)}$;

hence $$\int_x \sqrt{\left(\frac{1+x}{1-x}\right)} = \sin^{-1}x - \sqrt{(1-x^2)}.$$

(*c*) *Required* $\int_x \sqrt{(a^2-x^2)}$.

$$\sqrt{(a^2-x^2)} = \frac{a^2-x^2}{\sqrt{(a^2-x^2)}} = \frac{a^2}{\sqrt{(a^2-x^2)}} - \frac{x^2}{\sqrt{(a^2-x^2)}},$$

and $$\int_x \sqrt{(a^2-x^2)} = a^2 \int_x \frac{1}{\sqrt{(a^2-x^2)}} - \int_x \frac{x^2}{\sqrt{(a^2-x^2)}},$$

whence $$\int_x \sqrt{(a^2-x^2)} = a^2 \sin^{-1}\frac{x}{a} - \int_x \frac{x^2}{\sqrt{(a^2-x^2)}},\text{ by Art. 77;}$$

but $$\int_x \sqrt{(a^2-x^2)} = x\sqrt{(a^2-x^2)} + \int_x \frac{x^2}{\sqrt{(a^2-x^2)}},$$

by integration by parts, if we let

$$u = \sqrt{(a^2-x^2)} \text{ and } D_x v = 1.$$

Adding our two equations, we have

$$2\int_x \sqrt{(a^2-x^2)} = x\sqrt{(a^2-x^2)} + a^2 \sin^{-1}\frac{x}{a};$$

and $$\therefore \int_x \sqrt{(a^2-x^2)} = \frac{1}{2}\left(x\sqrt{a^2-x^2} + a^2 \sin^{-1}\frac{x}{a}\right).$$

EXAMPLES.

Find

(1) $\int_x \sqrt{(x^2+a^2)}$.

Ans. $\frac{1}{2}[x\sqrt{(x^2+a^2)} + a^2 \log(x+\sqrt{x^2+a^2})]$.

(2) $\int_x \sqrt{(x^2-a^2)}$.

Ans. $\frac{1}{2}[x\sqrt{(x^2-a^2)} - a^2 \log(x+\sqrt{x^2-a^2})]$.

Applications.

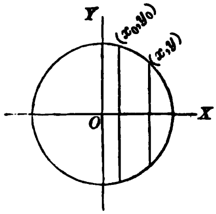
83. *To find the area of a segment of a circle.*

Let the equation of the circle be

$$x^2+y^2=a^2,$$

and let the required segment be cut off by the double ordinates through (x_0,y_0) and (x,y). Then the required area

$$A = 2\int_x y + C.$$



From the equation of the circle,

$$y = \sqrt{(a^2 - x^2)},$$

hence $$A = 2\int_x \sqrt{(a^2 - x^2)} + C;$$

and therefore, by Art. 82 (c),

$$A = x\sqrt{(a^2 - x^2)} + a^2 \sin^{-1}\frac{x}{a} + C.$$

As the area is measured from the ordinate y_0 to the ordinate y,

$$A = 0 \text{ when } x = x_0;$$

therefore $$0 = x_0\sqrt{(a^2 - x_0^2)} + a^2 \sin^{-1}\frac{x_0}{a} + C,$$

$$C = -x_0\sqrt{(a^2 - x_0^2)} - a^2 \sin^{-1}\frac{x_0}{a},$$

and we have

$$A = x\sqrt{(a^2 - x^2)} + a^2 \sin^{-1}\frac{x}{a} - x_0\sqrt{(a^2 - x_0^2)} - a^2 \sin^{-1}\frac{x_0}{a}.$$

If $x_0 = 0$, and *the segment begins with the axis of* Y,

$$A = x\sqrt{(a^2 - x^2)} + a^2 \sin^{-1}\frac{x}{a}.$$

If, at the same time, $x = a$, *the segment becomes a semicircle*, and

$$A = a\sqrt{(a^2 - a^2)} + a^2 \sin^{-1}\frac{a}{a} = \frac{\pi a^2}{2}.$$

The area of the whole circle is πa^2.

EXAMPLES.

(1) Show that, in the case of an ellipse,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

the area of a segment beginning with any ordinate y_0 is

$$A=\frac{b}{a}\left[x\sqrt{(a^2-x^2)}+a^2\sin^{-1}\frac{x}{a}-x_0\sqrt{(a^2-x_0^2)}-a^2\sin^{-1}\frac{x}{a}\right].$$

That if the segment begins with the minor axis,

$$A=\frac{b}{a}\left[x\sqrt{(a^2-x^2)}+a^2\sin^{-1}\frac{x}{a}\right].$$

That the area of the whole ellipse is πab.

(2) The area of a segment of the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

is
$$A=x\sqrt{(x^2-a^2)}-a^2\log(x+\sqrt{x^2-a^2})$$
$$-x_0\sqrt{(x_0^2-a^2)}+a^2\log(x_0+\sqrt{x_0^2-a^2}).$$

If $x_0=a$, and the segment begins at the vertex,

$$A=x\sqrt{(x^2-a^2)}-a^2\log(x+\sqrt{x_2-a_2})+a^2\log a.$$

84. *To find the length of any arc of a circle*, the coördinates of its extremities being (x_0,y_0) and (x,y).

By Art. 52, $\quad s=\int_x\sqrt{[1+(D_xy)^2]}.$

From the equation of the circle,

$$x^2+y^2=a^2,$$

we have $$2x + 2yD_xy = 0,$$

$$D_xy = -\frac{x}{y},$$

$$1 + (D_xy)^2 = \frac{x^2 + y^2}{y^2} = \frac{a^2}{y^2},$$

$$s = \int_x \frac{a}{y} = a\int_x \frac{1}{\sqrt{(a^2 - x^2)}} = a\sin^{-1}\frac{x}{a} + C. \quad \text{(Art. 77.)}$$

When $x = x_0,\qquad s = 0;$

hence $$0 = a\sin^{-1}\frac{x_0}{a} + C,$$

$$C = -a\sin^{-1}\frac{x_0}{a},$$

and $$s = a\left(\sin^{-1}\frac{x}{a} - \sin^{-1}\frac{x_0}{a}\right).$$

If $x_0 = 0$, and *the arc is measured from the highest point of the circle,* $$s = a\sin^{-1}\frac{x}{a}.$$

If the arc is a quadrant, $x = a$,

$$s = a\sin^{-1}(1) = \frac{\pi a}{2},$$

and the whole circumference $= 2\pi a$.

85. *To find the length of an arc of the parabola* $y^2 = 2mx$.

We have $$2yD_xy = 2m;$$

$$D_xy = \frac{m}{y};$$

$$\sqrt{[1 + (D_xy)^2]} = \sqrt{\left(\frac{m^2 + y^2}{y^2}\right)} = \frac{1}{y}\sqrt{(m^2 + y^2)};$$

$$s = \int_x \left[\frac{1}{y}\sqrt{(m^2+y^2)}\right] = \int_y \left[\frac{1}{y}\sqrt{(m^2+y^2)}\ D_y x\right];$$

$$D_y x = \frac{1}{D_x y} = \frac{y}{m}, \qquad \text{by Art. 73};$$

$$s = \frac{1}{m}\int_y \sqrt{m^2+y^2} = \frac{1}{2m}\left[y\sqrt{m^2+y^2} + m^2 \log(y+\sqrt{m^2+y^2})\right] + C,$$

by Art. 82, Ex. 1.

If the arc is measured from the vertex,

$$s = 0 \text{ when } y = 0;$$

$$0 = \frac{1}{2m}(m^2 \log m) + C,$$

$$C = -\frac{1}{2} m \log m,$$

and
$$s = \frac{1}{2}\left[\frac{y\sqrt{(m^2+y^2)}}{m} + m \log \frac{y+\sqrt{(m^2+y^2)}}{m}\right].$$

EXAMPLE.

Find the length of the arc of the curve $x^3 = 27y^2$ included between the origin and the point whose abscissa is 15.

Ans. 19.